Topological Vector Space Notes

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Contents

Chapter 1

Preliminaries

Throughout these notes we will fix a field $F = \mathbb{R}$ or $\mathbb{C}$ we will work with a vector space over this field.

Definition 1. Let X be a vector space $A \subset X$. A is *absorbing* if for all $x \in X$ there is $r > 0$ s.t. for all $s \in [0, r]$ one has $rx \in A$.

Question 1. If X is a vector space and $A \subset X$ does it follow that there is a convex subset of A that contains 0?

Definition 2. Let X be a vector space and $A \subset X$. A is *balanced* if for all $x \in A$, $\lambda \in \mathbb{R}$ s.t. $|\lambda| \leq 1$ one has $\lambda x \in A$.

For the case $F = \mathbb{R}$ being balanced means that for all $x \in A$ one has $[-x, x] \subset A$. On the other hand if $F = \mathbb{C}$ then being balanced means that for all $x \in A$ one has $\overline{D}x \subset A$ where $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Definition 3. Let X be a vector space and $A \subset X$. We say that A is *convex* if for all $x, y \in A$ one has $\lambda x + (1 - \lambda)y \in A$ for $0 \leq \lambda \leq 1$.

Definition 4. Let X be a vector space and $A \subset X$. The set of all convex combinations of A is called the convex hull of A and denote it by $\operatorname{conv} A$.

We can write

$$\operatorname{conv} A = \{\sum_{i=1}^{n} \lambda_i x_i : x_i \in A \text{ and } \lambda_i \geq 0 \text{ such that } \sum_{i=1}^{n} \lambda_i = 1\}.$$

Definition 5. Let X be a vector space and $A \subset X$. A is called *absolutely convex* if for all $x, y \in A$ and $\lambda, \mu \in \mathbb{R}$ with $|\lambda| + |\mu| = 1$ one has $\lambda x + \mu y \in A$.

Exercise 1. A set $A \subset X$ is absolutely convex iff it is convex and balanced iff it is convex and origin symmetric.

Definition 6. Let Ω be a set. A topology τ on Ω is a collection of subset sof Ω which is closed under arbitrary unions, finite intersections, and contains $\varnothing$ and Ω.

Elements of τ are called open sets. A set $B \subset A$ is called closed if A^c is open.

Definition 7. Let Ω be a topological space and $x \in \Omega$. Let $A \subset \Omega$. If there exists and open set N such that $x \in N \subset A$ we say that A is a neighbourhood of x and x is an interior point of A.

We denote by N_x the set of all neighbourhoods of x. $N \in N_x$ means that N is a neighbourhood of x. A set A is open iff for every $x \in A$ there exists $N \in N_x$ such that $N \subset A$.

Definition 8. The interior of A, denoted by $\operatorname{int} A$ is the set of interior points of A which is the largest open set contained in A.

Definition 9. The closure of A, denoted by $\overline{A}$ is the set of all points x such that for all $B \in N_x$ one has that $B \cap A$ is non-empty. $\overline{A}$ is the smallest closed set containing A.

We note that $\operatorname{int} A^c = (\overline{A})^c$ and then define the boundary of A to be $\partial A = \overline{A} - \operatorname{int} A$.

Definition 10. Let $x \in \Omega$. A subset B of N_x is called a base of neighbourhoods if for all $U \in N_x$ if there exists $V \in B$ such that $V \subset U$.

One has that $U \in N_x$ iff U is subset of some $V \in B$. A base is not unique. A set A is open iff for all $x \in A$, x has a base neighbourhood which is contained in A.

Definition 11. A topology is first countable if there exists a countable base at every point.

Definition 12. Γ is a directed set if Γ is equipped with a relation $\leq$ which is reflexive, transitive, and every two elements in Γ have a common successor. A net $x : \Gamma \longrightarrow \Omega$ is a function from a directed set to Ω. We write $x_\alpha = x(\alpha)$.

Reminder, a sequence is just a net where $\Gamma = \mathbb{N}$.

Definition 13. Let $(x_\alpha)_{\alpha \in \Gamma}$ be a net and α_0 be a fixed index. A tail of the net $(x_\alpha)_{\alpha \geq \alpha_0}$.

Definition 14. Let Ω be a topological space and $x \in \Omega$. Let (x_α) be a net in Ω. We say that (x_α) converges to x and write $x_\alpha \longrightarrow x$ if every neighbourhood of x contains a tail of (x_α).

We can recover neighbourhoods from convergence. A set U is a neighbourhood of $x \in \Omega$ if every net converging to x has a tail contained in U. If the topology is first countable then it suffices to consider sequences.

Example 1. Let Ω be a set and d be a metric on Ω. We say that a subset A of τ is open if for all $x \in A$ there is $r > 0$ such that $B(x, r) \subset A$.

This is a topology, called the metric topology. The open balls are a base, but in particular it is first countable since we can get a countable sub-base of balls.

Definition 15. If τ and σ are two topologies on Ω, we say that τ is weaker than σ if $\tau \subset \sigma$.

For convergence, if $x_\alpha \longrightarrow_\sigma x$ then $x_\alpha \longrightarrow_\tau x$.

Definition 16. Let Ω be a topological space and $A \subset \Omega$. The induced topology on A is defined by $\tau_A = \{A \cap O : O \in \tau\}$.

Convergence is preserved when passing to the induced topology. That is, when we have $A \subset \Omega$ and $(x_\alpha) \subset A$ then $(x_\alpha) \longrightarrow_{\tau_A} x$ iff $(x_\alpha) \longrightarrow_\tau x$.

Definition 17. Let Ω be a topological space and $x \in \Omega$. (x_α) a net in Ω. x is an accumulation point of (x_α) if for all $U \subset N_x$ and for all α there is $\beta \geq \alpha$ such that $x_\beta \in U$.

Having an accumulation point means that every tail of (x_α) has a subnet that conveges to x.

Definition 18. A topological space Ω is compact if every open cover of Ω has a finite subcover. Equivalently, every net in Ω has an accumulation point.

If Ω is a topological space and $A \subset \Omega$ we say that A is compact if it is compact with respect to the induced topology.

Definition 19. A topological space Ω is Hausdorff if for all $x \neq y$ there is $V \in N_X$ and $U \in N_y$ such that $U \cap V = \varnothing$. Equivalenty, nets have unique limits.

This means that if $x_\alpha \longrightarrow x$ and $x_\alpha \longrightarrow y$ then $x = y$.

Definition 20. Let $f : \Omega_1 \longrightarrow \Omega_2$ where Ω_1 and Ω_2 are topological spaces. f is continuous if for all open sets $U \subset \Omega_2$ one has $f^{-1}(U)$ is open in Ω_1. Equivalenty, for all $x \in \Omega_1$ and for all $V \in N_{f(x)}$ there is $U \in x$ such that $f(U) \subset V$. Equivalenty, if $x_\alpha \longrightarrow x$ in Ω_1 then $f(x_\alpha) \longrightarrow f(x)$ in Ω_2.

If $f : \Omega_1 \longrightarrow \Omega_2$ is contiunuous and $K \subset \Omega$ is a compact set, then $f(K)$ is again compact.

Definition 21. Let Ω_1 and Ω_2 be two topological spaces. We define the product topology on $\Omega_1 \times \Omega_2$ by $(x_\alpha, y_\alpha) \longrightarrow (x, y)$ in $\Omega_1 \times \Omega_2$ if $x_\alpha \longrightarrow x$ in Ω_1 and $y_\alpha \longrightarrow y$ in Ω_2

We call this topology the topology of coordinate-wise convergence. Equivalently we can consider the following construction. Define maps $p_1 : \Omega_1 \times \Omega_2 \longrightarrow \Omega_1$ and $p_2 : \Omega_1 \times \Omega_2 \longrightarrow \Omega_2$ to be projections. The product topology is the weakest topology on $\Omega_1 \times \Omega_2$ whcih makes btoh p_1 and p_2 continuous (this exists by either considering the intersection of all topologies or by Zorn's lemma). For $x \in \Omega_1$ and $y \in \Omega_2$ take $U \in N_x$ and $v \in N_y$. Then $\{U \times V : U \in N_x,\, v \in N_y\}$ is a base of neighbourhoods for the product topology for (x, y).

Definition 22. Let (Ω_α) be a family of topological spaces. Consider the Cartestian product

$$\Omega = \prod_{\alpha\in\Gamma} \Omega_\alpha = \{f : \Gamma \longrightarrow \bigcup_{\alpha\in\Gamma} \Omega_\alpha : \forall \alpha \in \Omega\, f(\alpha) \in \Omega_\alpha\}.$$

For a net (f_α) in Ω and $f \in \Omega$ we sauy that $f_\alpha \longrightarrow f$ if for all $\delta \in \Gamma$ one has $f_\alpha(\delta) \longrightarrow f(\delta)$.

This is the topology of point-wise convergence. Equivalently this the the least topology that makes all coordinate projections continuous. For base neighbourhoods. Fix $f \in \Omega$ and a finite collection of $\gamma_1, \ldots, \gamma_n$. For each $i = 1, \ldots, n$ pick some $U_i \in N_{f(x)}$ in Ω_{γ_i}. Put $W = \{g \in \Omega : \forall i = 1, \ldots, n\, g(\delta_i) \in U_i\}$. Set of this form, form a base of the product topology.

Theorem 1. *The product of a family of compact topological spaces is compact.*

Proof. Omitted. □

1.1 Topological Vector Spaces

Definition 23. X is a topological vector space if it is a vector and and a topological space such that addition and scalar multiplication are continuous. That is, addition $X \times X \longrightarrow X$ and scalar multiplcation $F \times X \longrightarrow X$ are continuous (where F has the usual topology).

In the language of nets, if $x_\alpha \longrightarrow x$ and $y_\alpha \longrightarrow y$ then $x_\alpha + y_\alpha \longrightarrow x + y$. Similarly if $x_\alpha \longrightarrow x$ and $\lambda_\alpha \longrightarrow \lambda$ then $\lambda_\alpha x_\alpha \longrightarrow \lambda x$. Let X be a topological vector space. Fix $y \in X$. Consider $f : X \longrightarrow X$ defined by $f(x) = x + a$, called the shift operator. By the definition, f is continuous and $f^{-1}(x) = x - a$ is also continuous. Thus, f is a homeomorphism. If U is a neighbourhood of x then $f(U) = U + a$ is a neighbourhood of $x + y$. Therefore, for every x one has $N_x = \{x + U : U \in N_0\}$, that is, neighbourhoods of any point are determined by the neighbourhoods at 0. This means that N_0 determines the underlying topology. In particular the topology is first countable iff N_0 has a countable base. Similarly for $\lambda \in F$ one has $x \longrightarrow \lambda x$ is a homeomorphism. If U is open then λU is open. If $U \in N_0$ then $\lambda U \in N_0$. This underlying topology is called a linear topology.

Exercise 2. X is Hausdorff iff $\bigcap N_0 = \{0\}$.

Exercise 3. The product of a family of topological vector spaces is again a topological vector space.

Example 2. A semi-normed space is a vector space equipped with a function $\rho : X \longrightarrow \mathbb{R}_{\geq 0}$ such that $\rho(\lambda x) = |\lambda|\, \rho(x)$, and $\rho(x + y) \leq \rho(x) + \rho(y)$. Such a function is called a semi-norm. If $\rho(x) = 0$ implies $x = 0$ then ρ is called a norm. (This means that the kernel of ρ is $\{0\}$). We prove that every semi-normed space is a topological vector space in the following Lemma.

First remember that we can define our topology of the semi-normed space two ways. Either through convergence, that is $x_\alpha \longrightarrow x$ if $\rho(x - x_\alpha) \longrightarrow 0$ in $\mathbb{R}$, or we can define the closed balls to be

$$B(x, r) = \{y \in X : \rho(x, y) \leq r\}$$

and then

$$\{B(x, r) : r > 0\}$$

is a base of the topology at x.

Lemma 2. *Every semi-normed space is a topological vector space, where the topology is generated by the open balls or by convergence.*

Proof. Suppose that $x_\alpha \longrightarrow x$ and $y_\alpha \longrightarrow y$. Then $\rho(x_\alpha - x) \longrightarrow 0$ and $\rho(y_\alpha - y) \longrightarrow 0$ and thus

$$\rho((x_\alpha + y_\alpha) - (x + y)) \leq \rho(x_\alpha - x) + \rho(y_\alpha - y) \longrightarrow 0,$$

hence $x_\alpha + y_\alpha \longrightarrow x + y$. Further if $\lambda_\alpha \longrightarrow \lambda$ and $x_\alpha \longrightarrow x$ then $\rho(x_\alpha - x) \longrightarrow$ which implies that

$$\rho(\lambda_\alpha x_\alpha - \lambda x) = \rho(\lambda_\alpha x_\alpha + -\lambda_\alpha x + \lambda_\alpha x - \lambda x) \leq \rho(\lambda_\alpha x_\alpha - \lambda_\alpha x) + \rho(\lambda_\alpha x - \lambda x) = |\lambda_\alpha| \, \rho(x_\alpha - x) + |\lambda_\alpha - \lambda| \, \rho(x) \longrightarrow 0,$$

so $\lambda_\alpha x_\alpha \longrightarrow \lambda x$. $\square$

Now we have the following fact:

Lemma 3. $\ker \rho$ *is a subspace.*

Proof. if $x, y \in \ker \rho$ then $\rho(x) = 0$, $\rho(y) = 0$. Then $\rho(x + y) \leq \rho(x) + \rho(y) = 0$ so $\rho(x + y) = 0$, so $x + y \in \ker \rho$. Similarly if $x \in \ker \rho$ then $\lambda x \in \ker \rho$. $\square$

This means that $\ker \rho = B(0, 0) \subset B(0, r)$ for every $r \geq 0$ and so essentially balls are absolutely huge. One can write $\ker \rho = \bigcap_{r>0} B(0, r) = \bigcap N_0$. One has from this that a semi normed space is Hausdorff iff ρ is a norm. Now we will show how to construct a norm from a semi-norm. If X is a vector space then $X/\ker \rho$ is a vector space. For $x \in X$ one has $\overline{x} = x + \ker \rho$. One has then that ρ is constant on each equivalence class.

Proof. Suppose that $x \sim y$, then $x - y \in \ker \rho$ so $\rho(x - y) = 0$. Then $\rho(x) = \rho(x - y + y) \leq \rho(x - y) + \rho(y) = \rho(y)$. This implies that $rho(x) \leq \rho(y)$, by symmetry we conclude that $\rho(x) = \rho(y)$. $\square$

For an equivalence class $\overline{x}$ put $\|\overline{x}\| = \rho(x)$. This is well defined by the previous remark.

Exercise 4. $\|\cdot\|$ is a norm on $X/\ker \rho$.

Fix a vector space. For each linear operator $T : X \longrightarrow Y$ where Y is a normed space one can define a semi-norm on X via $\rho(x) = \|Tx\|$.

Exercise 5. This is a semi-norm on X.

Every semi-nrom arises this way. Let ρ be a semi-norm arising this way. Put $Y = X/\ker\rho$. This a normed space. Let $\pi : X \longrightarrow Y$ be the quotient map. Then for each $x \in X$ one has $\rho(x) = \|\overline{x}\| = \|Qx\|$. This a correspondence between semi-norms on X and linear operators $T : X \longrightarrow Y$.

Example 3. Let (Ω, A, μ) be a measure space. Define $L_0(\mu)$ to be the space of all (equivalence classes) of measurable functions. For a net (f_α) in $L_0(\mu)$ one has that $(f_\alpha) \longrightarrow_\mu f$ (convergence in measure) if for all $\varepsilon > 0$ there is α_0 such that for all $\alpha \geq \alpha_0$ then $\mu(|f - f_\alpha| > \varepsilon) < \varepsilon$. This convergence corresponds to a topology.

For every $\varepsilon > 0$ let $U_{f,\varepsilon} = \{g \in L_0(\mu) : \mu(|f - g| > \varepsilon) < \varepsilon\}$. These sets, as $\varepsilon > 0$, form a base of the neighbourhoods for f. This topology is linear and if $f_\alpha \longrightarrow_\mu f$ and $g_\alpha \longrightarrow_\mu g$ then $f_\alpha + g_\alpha \longrightarrow_\mu f + g$. If $f_\alpha \longrightarrow_\mu f$ and $\lambda_\alpha \longrightarrow \lambda$ then $\lambda_\alpha f_\alpha \longrightarrow_\mu \lambda f$.

Example 4. Almost everywhere convergence is not topological!

For every topological convergence, a sequence (x_n) convergence to x iff every subsequence (x_{n_r}) has a further subsequence $(x_{n_{r_l}})$ converging to x. In regards to Example 4 consider $L_0[0,1]$. For $n \in \mathbb{N}$ let $r_n = \sum_{k=1}^{n} \frac{1}{k}$. Then $0 \leq r_n \longrightarrow r_{n+1} \longrightarrow \infty$. Let $x_n = \chi_{A_n}$ where $A_n = [r_n, r_{n+1}] \mod 1$. This is called the "walking ghost" or "type-writer sequence". Then we note that (x_n) does not converge to 0 a.e., but on the other hand every subsequence of (x_n) has a further subsequence that converges to 0 a.e.

We remark on point-wise convergence. Consider functions on $[0,1]$, $F[0,1] = \mathbb{R}^{[0,1]}$. We equip F with point-wise convergence. The base neighbourhoods are given by choosing $t_1, \ldots t_n \in [0,1]$ and considering an ε neighbourhood of $f(t_i)$ where $f \in F[0,1]$. Consider the set of all $g \in F[0,1]$ such that $|f(t_i) - g(t_i)| < \varepsilon$ for all $i = 1, \ldots, n$. Denote this set by $V_{\varepsilon,t_1,\ldots,t_n}$. Then $V_{\varepsilon,t_1,\ldots,t_n}$ is a base neighbourhood for f.

Example 5. Let X be a Banach space. A net $(x_\alpha) \in X$ converges weakly to x if $f(x_\alpha) \longrightarrow x$ for every $f \in X^*$. This convergence corresponds to the weak topology w on X.

This topology is the least topology that makes all $f \in X^*$ continuous. Then (X, w) is a topological vector space. If $x_\alpha \longrightarrow x$ and $y_\alpha \longrightarrow y$ then for every $f \in X^*$, then $f(x_\alpha) \longrightarrow f(x)$ and $f(y_\alpha) \longrightarrow f(y)$, then $f(x_\alpha + y_\alpha) = f(x_\alpha) + f(y_\alpha) \longrightarrow f(x) + f(y) = f(x + y)$. Similarly, if $x_\alpha \longrightarrow x$ and $\lambda_\alpha \longrightarrow \lambda \in F$ then $\lambda_\alpha x_\alpha \longrightarrow \lambda x$.

One clearly has that the weak topology on X is weaker than the norm topology on X. If $x_\alpha \longrightarrow_{\|\cdot\|} 0$ then $\|x_\alpha\| \longrightarrow 0$ then for each $f \in X^*$ one has $|f(x_\alpha)| \leq \|f\| \, \|x\| \longrightarrow 0$, so $f(x_\alpha) \longrightarrow 0$ so $x_\alpha \longrightarrow 0$.

Example 6. Let $X = \ell_p$, $1 < p < \infty$ and consider $X^* = \ell_q$ where $q = p^*$. Consider (e_n) the standard unit basis of X. Then $\|e_n\| = 1$ for all n, so (e_n)

does not converge to zero in norm. But we claim that $e_n \longrightarrow_w 0$. If $f \in X^* = \ell_q$. Then $f = (f_i)$ and $\|f\|_{\ell_q} = \left(\sum_{i=1}^{\infty} |f_i|^q\right)^{1/q}$, so $\sum |f_i|^q$ converges, so $f_i \longrightarrow 0$, but $f(e_n) = f_n \longrightarrow 0$.

Let X be a Banach space. Consider X^*. This is again a Banach space. On X^* we have the norm topology and the weak topology. For a net $(f_\alpha) \in X^*$ we have that $f_\alpha \longrightarrow_w f$ if $\forall \varphi \in X^{**}$ one has $\varphi(f_\alpha) \longrightarrow \varphi(f)$.

Definition 24. We say that (f_α) converges to f in weak* if $f_\alpha(x) \longrightarrow f(x)$ for all $x \in X$.

Recall we have the isometric embedding of $X \longrightarrow X^{**}$ given by $j(x) = \hat{x}$ where $\hat{x}(f) = f(x)$. weak convergence is witnessed by all elements of X^{**}. weak* convergence is only witnessed by those elements of X^{**} which come from X. So this implies that if $f_\alpha \longrightarrow_w f$ then $f_\alpha \longrightarrow_{w^*} f$. One has that norm convergence implies weak convergence which implies weak* convergence. If X is reflexive then our inclusion map is onto, and thus $X = X^{**}$, so weak = weak* (on X^*).

Example 7. Let $X = c_0$. Then $X^* = \ell_1$ and $X^{**} = \ell_\infty$. Let (f_n) be the standard unit basis in ℓ_1. Then (f_n) does not converge to zero weakly because $1 \in \ell_\infty$ has $1(f_n) = 1 \not\longrightarrow 0$. But each f_n converges to zero in weak*. For each $x \in c_0$ one has $f_n(x) =$ (the n-th component of x)$\longrightarrow 0$ because $x \in c_0$, so $w \neq w^*$.

Each $f \in X^*$ is a function from X to F. One has $X^* \subset F^X$. We have w^* convergence on X^* is the restriction to X^* of the pointwise convergence on F^X. For each $\varepsilon > 0$ fix a finite set $x_1, \ldots, x_n \in X$. Then the sets $V_{\varepsilon, x_1, \ldots, x_n} = \{f \in X^* : |f(x_i)| < \varepsilon \, \forall i\}$ form a base of neighbours of zero for (X^*, w^*). For weak topology, we can just flip vectors with functions. Fix $\varepsilon > 0$ and $f_1, \ldots, f_n \in X^*$. Then $U_{\varepsilon, f_1, \ldots, f_n} = \{x \in X : |f_i(x)| < \varepsilon \, \forall i\}$ form a base of neighbourhoods of zero for (X, w).

Lemma 4. *Let X be a Banach space. If $x_n \longrightarrow_w 0$ in X then (x_n) is norm bounded. If $(f_n) \longrightarrow_{w^*} 0$ in X^* then (f_n) is norm bounded.*

Proof. Suppose that $f_n \longrightarrow_{w^*} 0$. Then for each $x \in X$ we have $f_n(x) \longrightarrow 0$, so $(f_n(x))$ is bounded for all $x \in X$. Then by the Uniform Boundedness Principle we conclude that (f_n) is norm bounded.

Now suppose that $x_n \longrightarrow_w 0$. For each $f \in X^*$ one has $f(x_n) \longrightarrow 0$ so $(f(x_n))$ is bounded. This is precisely $(\hat{x}_n(f))$ and thus $(\hat{x}_n)$ is point-wise bounded. Thus by Uniform Boundedness Principle one has $(\hat{x}_n)$ is norm bounded in X^{**}. Thus there is $M > 0$ such that for all n $M \geq \|\hat{x}_n\| = \|x_n\|$. Thus (x_n) is bounded in X. $\square$

Theorem 5. *Let X be a Banach space. The weak topology and the norm topology agree iff* $\dim X < \infty$.

Proof. Suppose $\dim X < \infty$. All norms on X are equivalent, so without loss of generality, we can assume $X = \ell_1^n$. If $x_\alpha \longrightarrow_w 0$, then x_α converges to zero coordinate-wise (because coordinates are linear functionals). In ℓ_1^n, coordinate convergence implies norm convergence. On the other hand, if the two topologies agree then every base neighbourhood must contain a base neighbourhood of the other. Thus, B_X, which is a zero neighbourhood for the norm topology must contain $V_{\varepsilon,t_1,\dots,t_n} \supset \{x \in X : f_i(x) = 0 \,\forall i = 1,\dots,n\} = \bigcap_{i=1}^n \ker f$, which is a subspace of dimension at most n. This means that

$$\bigcap \ker f_i = \{0\},$$

so $\dim X \leq n$. $\square$

Example 8. Take $X = \ell_2$. Take $A = \{\sqrt{n}e_n : n \in \mathbb{N}\}$. Then A meets $V_{\varepsilon,f_1,\dots,f_m}$ for any $\varepsilon > 0$, $f_i \in X^*$, but no sequence of A weakly converges to zero.

This shows that A contains a net which weakly converges to zero, but no sequences, so A the weak topology is not first countable. We need nets. Sequences are not enough.

Example 9. Let X, and Y be Banach spaces. $L(X,Y)$ is again a Banach space. For a net (T_α), we say that $T_\alpha \longrightarrow_S T$ if $T_\alpha x \longrightarrow_{\|\cdot\|} Tx$ in Y. We say that $T_\alpha \longrightarrow_W T$ if $T_\alpha x \longrightarrow_w Tx$ in Y. Then for a net $(T_\alpha) \in L(X,Y^*)$, we say that $T_\alpha \longrightarrow_{W^*} T$ if $T_\alpha x \longrightarrow_{w^*} Tx$ in Y^*. These are the strong, weak, and weak* operator topologies. All of these are linear topologies. What is a base of neighbourhoods of each?

Proof of above for strong operator topology. Suppose that $(T_\alpha), (S_\alpha) \subset L(X,Y)$ are two nets which converge strongly. Then for all $x \in X$ one has

$$\|T_\alpha x - Tx\| \longrightarrow 0,$$

and

$$\|S_\alpha x - Sx\| \longrightarrow 0.$$

Then

$$\|(T_\alpha + S_\alpha)x - (T+S)x\| \leq \|T_\alpha x - Tx\| + \|S_\alpha x - Sx\| \longrightarrow 0.$$

Then if $\lambda_\alpha \longrightarrow \lambda$ we have

$$\|\lambda_\alpha x T_\alpha - \lambda Tx\| = \|\lambda_\alpha T_\alpha x + \lambda_\alpha Tx - \lambda_\alpha Tx - \lambda Tx\| \leq |\lambda_\alpha|\,\|T_\alpha x - Tx\| + |\lambda_\alpha - \lambda)|\,\|Tx\| \longrightarrow 0.$$

A base of neighbourhoods is given by

$$V_{\varepsilon,x_1,\dots,x_n} = \{T \in L(X,Y) : \|Tx_i\| < \varepsilon \forall i\}.$$

$\square$

Now we return back to TVS.

Lemma 6. *Let $U \in N_0$. Then U is absorbing.*

Proof. Fix $x \in X$. Then $\lim_{\lambda \longrightarrow 0} \lambda x = 0$. Thus, there is $\varepsilon > 0$ such that for all λ with $|\lambda| \leq \varepsilon$ we have $\lambda x \in U$. In particular if $0 \leq \mu \leq \varepsilon$, then $\mu x \in U$. Thus U is absorbing. $\square$

Lemma 7. *Let $U \in N_0$. Then there exists $V \in N_0$ such that $V + V \subset U$. (This is essentially continuity of addition).*

Proof. If U is in N_0 then by continuity of addition there exists $W_1, W_2 \in N_0$ such that $W_1 + W_2 \subset U$. Take $V = W_1 \cap W_2$. $\square$

Lemma 8. *If A is balanced then so is $\overline{A}$.*

Proof. Let $x \in \overline{A}$ and $\lambda \in F$ with $|\lambda| \leq 1$. We want $\lambda x \in A$. Since $x \in \overline{A}$ there exists a net (x_α) of elements in A with $x_\alpha \longrightarrow x$. Then since A is balanced, $\lambda x_\alpha \in A$ and $\lambda x_\alpha \longrightarrow \lambda x \in \overline{A}$. $\square$

Note that if A is balanced, then A° is not always balanced, but $\{0\} \cup A^\circ$ is balanced.

Lemma 9. *Let X be a TVS. Then N_0 has a base of closed balanced sets.*

Proof. It suffices to show that all closed balanced sets in N_0 form a base of neighbourhoods of N_0. That is, every neighbourhood contains a closed balanced neighbourhood. Since scalar multiplication is continuous, there exists $\varepsilon > 0$ and $W \in N_0$ such that $\lambda x \in U$ when $|\lambda| \leq \varepsilon$, and $x \in W$. Put $V = \{\lambda x : |\lambda| \leq \varepsilon, x \in W\}$. Then $V \supset \varepsilon W \in N_0$. V is balanced by definition. Let $U \in N_0$, find $W \in N_0$ such that $W + W \subset U$. Then find a balanced V s.t. $V \in N_0$ and $V \subset W$. Then $\overline{V}$ is closed and balanced. All is left to show is that $\overline{V} \subset U$. Let $x \in \overline{V}$ and find a net $(x_\alpha) \subset V$ with $x_\alpha \longrightarrow x \in V$. Thus $x_\alpha - x \longrightarrow 0$. Since $V \in N_0$ there is a tail of x_α with $(x_\alpha)_{\alpha \geq \alpha_0} \subset V$. Then $x = x - x_\alpha + x_\alpha \in V + W \subset W + W \subset U$. $\square$

Corollary 1. *A TVS is Hausdorff iff $\bigcap N_0 = \{0\}$.*

Proof. Suppose that X is Hausdorff and that $x \neq 0 \in X$. Then there exists $U \in N_x$, $V \in N_0$ such that $U \cap V = \varnothing$, so $x \notin V$, so $x \notin \bigcap N_0$. In other words $\bigcap N_0 = \{0\}$.

Suppose now that $\bigcap N_0 = \{0\}$. Let $x \neq y$. Take $z = x - y \neq 0$, so $z \notin \bigcap N_0 \subset V$ for all $V \in N_0$. For all $U \in N_0$ there is $W \in N_0$ such that $W + W \subset U$. There exists $V \subset W$ such that V is balanced, $V \subset W$, so $V + V \subset U$. We claim that $(x + V) \cap (y + V) = \varnothing$. If not, then there is $a \in X$ such that $a = x + v_1 = y + v_2$, for $v_1, v_2 \in V$. so $a - x, a - y \in V$. Since V is balanced, $x - a \in V$. Then $z = x - a + a - y \in V + V \subset U$. A contradiction. $\square$

Theorem 10. *A topology on a vector space is linear iff for all $x, y \in X$, $U \in N_x$ and $0 \neq \lambda \in F$, one has*

$$U + y \in N_{x+y}, \lambda U \in N_{\lambda x}.$$

$$\forall U \in N_0,\ U \text{ is absorbing}$$

$$\forall U \in N_0, \exists V \in N_0, V \subset U \text{ and } V \text{ is balanced.}$$

$$\forall U \in N_0, \exists V \in N_0, V + V \subset U.$$

Proof. See Dhmitri's notes. □

Theorem 11. *A TVS X is metrizable iff it is first countable and Hausdorff.*

Proof. See Dhmitri's notes. □

Theorem 12 (Hahn-Banach Theorem)**.** *Let E be a vector space over $\mathbb{R}$. Let $\rho : E \longrightarrow [0, \infty)$ be a function satisfying $\rho(\lambda x) = \lambda\rho(x)$ for $\lambda \geq 0$ and $\rho(x+y) \leq \rho(x) + \rho(y)$. Let X be a vector subspace of E and $f : X \longrightarrow \mathbb{R}$ be a linear functional such that for all $x \in X : f(x) \leq \rho(x)$. Then f extends to a linear functional $F : E \longrightarrow \mathbb{R}$ such that for all $x \in E$ one has $f(x) \leq \rho(x)$.*

Proof. Math 516. □

Note, if ρ is indeed a seminorm, that is $\rho(\lambda x) = \lambda\rho(x)$ for all $\lambda \in \mathbb{R}$ then

$$\pm F(x) = F(\pm x) \leq \rho(\pm x) = \rho(x),$$

so $|f(x)| \leq \rho(x)$.

Theorem 13 (Hahn Banach for Locally Convex Topological Vector Spaces)**.** *Let X be a LCTVS, $Y \subset X$ a subspace. $f \in Y^*$ a continuous linear functional. Then f extends uniquely to a continuous linear functional $F \in X^*$.*

Proof. Consider first $k = \mathbb{R}$. Since f is continuous on Y, $f(0) = 0$, there exists a neighbourhood V_0 of zero in Y such that $f(V_0) \subset (-1, 1)$. Then $V_0 = V \cap Y$ for some neighbourhood of zero V in X. WLOG, V is convex and balanced (convex because X is locally convex). Then the Minkowski functional ρ_V is a seminorm on X. Let $y \in Y$. Then For every $s > P_V(y)$ one has $y/s \in V$ so $y/s \in V_0$. This means $|f(y/s)| < 1$ and so $f(y) < s$. Taking inf over $s > p_V(y)$ we obtain $f(y) \leq p_V(y)$. Then by Hahn-Banch we obtain a linear functional $F \in X^*$ such that for all $x \in X$ one has $|F(x)| \leq \rho(x)$. If $x_\alpha \longrightarrow 0$ then $p_V(x_\alpha) \longrightarrow 0$, so $F(x_\alpha) \longrightarrow 0$. For the case $F = \mathbb{C}$, we reduce to the case $F = \mathbb{R}$ by considering $\Re f$. □

Definition 25. For a TVS X we write

$$X^* = \{f : X \longrightarrow F : \ F \text{ continuous and linear}\}$$

Corollary 2. *Let X be a LCTVS and $Y \subset X$ a closed subspace and $a \notin Y$. Then there is a continuous linear functional $f \in X^*$ such that f vanishes on Y and $f(a) = 1$.*

Proof. Same as 516. Take $Z = \operatorname{span}\{Y, a\}$ and define $f : Z \longrightarrow F$ via $f(y + \lambda a) = \lambda$. This is well defined by liner algebra. f vanishes on Y, $f(a) = 1$, f is continuous because $\ker f = Y$ is closed. By Hanh Banach for LCTVS, f extends to $\in X^*$ so F still vanishes on Y and $F(a) = f(a) = 1$. □

Corollary 3. *Let X be a LCTVS, $x \in X$ and $x \neq 0$. Then there is $f \in X^*$ such that $f(x) \neq 0$.*

Proof. Apply the previous corollary for $Y = \{0\}$. □

We remark that the kernel of f is generally not the same as the kernel of F. The kernel of F could be much bigger. In particular, this means that X^* is non-trivial. That is, LCTVS have large duals.

Example 10. Let $X = L_0[0,1]$ with the topology of convergence in measure. This is a linear topology. Yet, it is not locally convex (HW). Actually, there are no proper convex sets in N_0. It follows that $X^* = \{0\}$ because if $0 \neq \varphi \in X^*$ then the set $\{x : |\varphi(x)| < 1\}$ is a convex neighbourhood of zero. This is the unit ball of ρ where $\rho(x) = |\varphi(x)|$.

Example 11. $X = L_p[0,1]$, $0 < p < 1$. Then X is a topological vector space but $X^* = \{0\}$, so X is not locally convex.

Example 12. $X = L_2[0,1] \oplus L_{1/2}[0,1]$ is not locally convex. $X^* = L_2[0,1] \oplus \{0\}$, so X^* is not trivial.

Example 13. $X = \ell_p$ for $0 < p < 1$. Then X^* is non-trivial because every coordinate functional is in X^*. Yet, X is not locally convex.

Proof. Suppose that X is locally convex. Then B_X contains a convex $V \in N_0$. $\{1/nB_X : n \in \mathbb{N}\}$ is a base of the topology. Then there is $n \in \mathbb{N}$ such that $1/nB_X \subset V$. It follows that $(1/n)B_X \subset V \subset B_X$. For every $n \in \mathbb{N}$ we have that $e_k \in B_X$, so $(1/n)e_k \in (1/n)B_X$. Thus $(1/b)e_k \in V$ and thus $x = (1/n)\sum_{k=1}^{n}(1/n)e_k \in V$ and also lies in B_X. But $\left\|(1/n^2)\sum_{k=1}^{n} e_k\right\| \leq 1$ and $\|\sum_{k=1}^{n} e_k\| \leq n^2$. Then $\|(1,1,1,\ldots 0,\ldots)\| = n^{1/p} < n^2$. □

Now we discuss some things about seperation of convex sets. Let X be a TVS over $\mathbb{R}$. Let $A, B \subset X$ be two non-empty subsets of X and $f : X \longrightarrow \mathbb{R}$ a linear functional (usually $f \in X^*$). f seperates A from B if $\sup_{a\in A} f \leq \inf_{b\in B} f$. That is, there is $c \in \mathbb{R}$ such that $f_{|A} \leq c$ and $f_{|B} \geq c$. A and B need not have empty intersection. Note that if f seperates A from B then $-f$ seperates B from A.

We remark that the seperation makes sense over any vector space. Not just a TVS. In the case that $\sup_{a\in A} f < \inf_{b\in B} f$ we say that f strictly seperates A and B.

Theorem 14. *Let X be a LCTVS. Let C be a convex non-empty subset of X with non-empty interior such that $0 \notin C^\circ$. Then there is $f \in X^*$ such that $f\,|_C \geq 0$.*

Proof. Take $a \in C^\circ$. Put $K = a - C$. Then $a \notin K$, $0 \in K^\circ$. Find a neighbourhood $V \in N_0$, $V \subset K$. Without loss of generality V is balanced and convex. Since $V \subset K$, $\rho_K \leq \rho_V$. Since V is balanced, p_V is a seminorm and K is convex and absorbing so ρ_K is always finite. One has $\rho_K(\lambda x) = \lambda\rho_K(x)$ when $\lambda \geq 0$. Since K is convex, $\rho_K(x+y) \leq \rho_K(x) + \rho_K(y)$. Since $a \notin K$ one has $\rho_K(a) \geq 1$. Let $Y = \operatorname{span}\{a\}$. Define $f : Y \longrightarrow \mathbb{R}$ such that $f(\lambda a) = \lambda\rho_K(a)$ for $\lambda \in \mathbb{R}$. Then f is a linear functional on Y with the property that $f(\lambda a) \leq \rho_K(\lambda a)$. Then by Hahn Banach f extends to $F \in X^*$ such that for all $x \in X$ one has $F(x) \leq \rho_K(x) \leq \rho_V(x)$. Then one verifies that $|F(x)| \leq \rho_V(x)$, so F is continuous. If $x_\alpha \longrightarrow 0$ then $\rho_V(x_\alpha) \longrightarrow 0$ which means $F(x_\alpha) \longrightarrow 0$.

For every $x \in C$ we have $a - x \in K$ so $a \geq \rho_K(a-x) \geq F(a-x) = F(a) - F(x) = f(a) - F(x) = \rho_K(a) - F(x) \geq 1 - F(x)$. □

We remark that if X a TVS then possibly the above theorem works, we need V not be convex. Check this.

Corollary 4. *Let A and B be two disjoint non-empty convex sets in a locally convex topological vector space X (possibly not locally convex). Suppose that $\int A \neq \varnothing$. Then there is a linear functional $f : X \longrightarrow \mathbb{R}$ seperating A from B.*

Proof. Take $C = A - B$. Since A and B are convex, so is C. Since $\int A \neq \varnothing$ then $\int C \neq \varnothing$. Since $A \cap B = \varnothing$ we have $0 \notin C$. Then by the theorem there is $f \in X^*$ such that $f|_C \geq 0$ and thus for all $a \in A$ and $b \in B$ we have $f(a-b) \geq 0$, so $f(a) \geq f(b)$. This proves the seperation. $\square$

Suppose that A and B are loesd and bounded, non-empty and disjoint. Can we strictly seperate? NO. The counter example is as follows. Let $X = c_0$ indexed by $\mathbb{N}$. Take $A = \overline{\text{conv}}\{e_n\}$ and $B = \overline{\text{conv}}\,((1/n)e_0 + e_n)$. Then A and B are closed and bounded and convex. Both are closed and convex. Both are bounded. $A \subset B_X$ and $B \subset 2B_X$. A and B can not be strictly seperated because $((1/n)e_0 + e_n) - e_n = (1/n)e_0 \longrightarrow 0$. Also $A \cap B \neq \varnothing$.

Theorem 15. *Let X be a LCTVS, $A, B \subset X$ being non-empty, disjoint, convex, where A is closed and B is compact. Then there is $f \in X^*$ which strictly seperates A from B.*

Proof. Let $C = A - B$. As before, C is convex, closed, and $0 \notin C$. Then there is $V \in N_0$ such that $V \cap C = \varnothing$. Without loss of generality, V is convex and balanced. Then from the corollary we can find $f \in X^*$ such that f seperates V and C. Then $\lambda = \sup_V f \leq \inf_V f$. Since $0 \in V$ we see that $\lambda \geq f(0) = 0$, so $\lambda \geq 0$. I claim that $\lambda > 0$. If $\lambda = 0$ then for all $x \in V$ we have $\pm x \in V$ For all $a \in A$ and $b \in B$ we have $a - b \in C$ so $f(a-b)$ and thus $\pm f(x) = f(\pm x) \leq \lambda = 0$, so f vanishes on V and since V is absorbing f vanishes everywhere, so f is zero, which is a contradiction. Since $\lambda > 0$ we have for all $a \in A$, $b \in B$ we have $a - b \in C$ so $f(a-b) \geq 0$ and thus $\inf_{a \in A} f \geq \sup_{b \in B} f + \lambda$ but since $\lambda > 0$ we get $\inf_{a \in A} f > \sup_{b \in B} f$. $\square$

Recall from last time, that if X is a TVS, A and B are disjoint non-empty convex sets, $A^\circ \neq \varnothing$. Then A and B can be seperated by some $f \in X^*$. If X is a locally convex topological vector space where A and B are disjoint, non-empty sets, where A is closed and B is compact then A and B can be strictly seperated by some $f \in X^*$. Let $f \in X^*$. Then the set $\{f = c\} = \{x \in X : f(x) = c\}$ is called a hyperplane where $c \in F$ (or $\mathbb{R}$ when we do seperation theorems). The set $\{f \leq c\}$ is called a (closed) halfspace. These are closed convex sets. For a set A, by $\text{conv}A$ we denote the least (intersection of all) convex set which contains A, or the set of all convex combinations of elements of A. Then we write $\overline{\text{conv}}A$ to be the closed convex hull of A.

Lemma 16. *Let X be a LCTVS, and $A \subset X$. Then $\overline{\text{conv}}A$ is the intersection of all closed half spaces containing A.*

Proof. One has $\overline{\text{conv}}A \subset C$ trivially. Suppose $x \in C$ but $x \notin \overline{\text{conv}}A$. Then since $\{x\}$ is compact and $\overline{\text{conv}}A$ is closed, both are convex, we can strictly seperate them. There is $f \in X^*$ and $c \in \mathbb{R}$ such that $\overline{\text{conv}}A \subset \{f \leq c\}$, but $x \notin \{f \leq c\}$. □

Corollary 5. *Every closed convex set is the intersection of all closed convex half-spaces containing it.*

This means that the closed convex sets in X are determined precisely by the continuous functionals on it. So, if X is a vector and τ_1 and τ_2 are two locally convex topologies on X such that $(X, \tau_1)^* = (X, \tau_2)^*$, then τ_1 and τ_2 have the same closed convex sets. In particular, they have the same closed subspaces.

As an application, if X is a Banach space. By the homework, a linear functional $f : X \longrightarrow \mathbb{R}$ is norm continuous iff it is weakly continuous. Thus $X^* = (X, \|\cdot\|)^* = (X, w)^*$. This means a convex set is norm closed iff it is weakly closed.

We now move onto the next big topic, but first we discuss some motivation. In a vector space we have the concept of a Hamel basis B, and the idea is that is is a "small" set, but we can recover the whole space as the span of this set B. In a Banach space, we have the concept of a Schauder basis S. That is, a relatively "small" (countable) set such that its closed span of S is the whole space. Now, given a convex set C in a vector space or a Banach space, we would like to find a "small" subset A of C so that $C = \text{conv}A$ or in the case of a Banach space $C = \overline{\text{conv}(A)}$. For a polytope in $\mathbb{R}^n$ we can take A to be the boundary of P or just the vertices V, and then write $A = \text{conv}\partial A$ or $A = \text{conv}V$. We wish to determine these points, and so we naturally come into the discussion of an extreme point.

Definition 26. If X is a vector space and C is a convex set then we say that $x \in C$ is an extreme point if $a, b \in C$ and $\lambda \in (0, 1)$ with $\lambda a + (1 - \lambda)b = x$ then $a = b = x$.

We denote $\text{ext}(C)$ to be the set of all extreme points of C. If X is a TVS then $\text{ext}(C) \subset \partial C$. A subset E of C is an extremal subset of C if provided that $x, y \in C$ and $\lambda \in (0, 1)$ such that $\lambda x + (1 - \lambda)y \in E$ then $x, y \in E$. An example is a face of a polytope which satisfies this condition.

Theorem 17 (Krein-Millman Theorem)**.** *Let C be a convex compact set in a LCTVS, then $C = \overline{\text{convext}(C)}$.*

Proof. We first prove that $\text{ext}(C) \neq \varnothing$. We will use Zorn's Lemma. Let A be the collection of all closed convex extremal subsets of C. This collection is non-empty, because C is in this collection. Consider a decreasing chain $E_1 \supset E_2 \supset E_3 \supset \dots$ Every element is compact being a closed subset of C. Thus we have the finite intersection property. Thus by compactness the entire chain has a non-empty intersection. This intersection is again in A (skipped just verify intersection of extreme sets is extreme). Thus by Zorn's Lemma A

has a minimal element, say M. M is an extreme set, we will show that M is a singleton. Suppose not. Then we can find $a \neq b \in M$. Then we can find $f \in X^*$ such that $f(a) < f(b)$, so f is not constant on M. Since M is a closed subset of a compact set C, M is compact. So f attains a minimum on M. Let $m = \min_M f$. Let $M_0 = M \cap \{f = m\}$. Since f is not constant we have M_0 is properly contained in M. We will show that M_0 is an extremal subset of C leading to a contradiction of minimality. Let $x, y \in C$, $\lambda \in (0, 1)$ and consider $\lambda x + (1 - \lambda)y \in M_0$. Then $\lambda x + (1 - \lambda)y \in M$. Since M is an extreme set, $x, y \in M$. Since $\lambda x + (1 - \lambda)y \in M_0 \subset \{f = m\}$, we have $m = f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y) \geq \lambda m + (1 - \lambda)m = m$. So $f(x) = m$ and $f(y) = m$ and thus $x, y \in M_0$. The contradicts the minimality of M.

Now we prove the second part by contradiction. Suppose there exists $a \in C$ such that $a \notin \overline{\text{conv}}\text{exT}C$. Since $\{a\}$ is convex and compact, and $\overline{\text{conv}}\text{ext}(C)$ is a closed subset of C, convex and compact, we can strictly seperate them. There is $g \in X^*$ such that $g(a) \leq c < d \leq g \mid_{\overline{\text{conv}}\text{ext}(C)}$. Let $s = \min_C g$. $L = C \cap \{g = s\}$. Then $g_L \leq g(a) \leq c < d \leq g \mid_{\overline{\text{conv}}\text{ext}(C)}$. L is a convex closed subset of C, hence compact. By part 1, L has extreme points, say $b \in \text{ext}(L)$. The claim is that $b \in \text{ext}(C)$ (this will be a contradiction). Take $x, y \in C$, $\lambda \in (0, 1)$, such that $\lambda x + (1 - \lambda)y = b$. Since $b \in L$, $s = g(b) = g(\lambda x + (1 - \lambda)y) = \lambda g(x) + (1 - \lambda)g(y) \geq \lambda s + (1 - \lambda)s = s$, so $g(x) = g(y) = s$, so $x, y \in L$. Since $b \in \text{ext}(L)$ we have $x = y = b$. □

Example 14. Let $X = C[0, 1]$ over $\mathbb{R}$. Denote $C = \{f \in X : -1 \leq f \leq 1\} = B_X = [-1, 1]$. C is convex. What is $\text{ext}(C)$? Then $\pm 1 \in \text{ext}(C)$. There are not other extreme points. One verifies that C is not compact with respect to any Hausdorff locally convex topology.

Now we discuss dual pairs. Let X be a normed space. X^* the dual space. For $f \in X^*$ f acts on X by $x \longrightarrow f(x)$. Similarly $x \in X$ acts on X^* by $f \longrightarrow f(x)$.

Definition 27. A dual pair is a pair fo vector spaces X and Y equipped with a bilinear map $\langle \cdot, \cdot \rangle : X \times Y \longrightarrow F$ that seperates points of X and Y: For all $x \neq 0 \in X$ there is $y \in Y$ such that $\langle x, y \rangle \neq 0$. Similarly for all $y \neq 0 \in Y$ there is $x \in X$ such that $\langle x, y \rangle \neq 0$.We write $\langle X, Y \rangle$. This definition is symmetric under the map $(y, x) \longrightarrow \langle x, y \rangle$.

Example 15. Let X be a normed spae. Then X and X^* are dual pairs. One has $\langle x, f \rangle = \langle f, x \rangle = f(x)$.

Example 16. Similarly, X^* and X^** are dual pairs via $\langle f, \xi \rangle = \xi(f)$.

Example 17. $L_p(\mu)$ and $L_q(\mu)$ are dual pairs under $\langle f, g \rangle = \int fg \, d\mu$, where $1/p + 1/q = 1$

Let $\langle X, Y \rangle$, be a dual pair. Each $y \in Y$ defines a linear functional on X via $x \longrightarrow \langle x, y \rangle$. Similarly, every $x \in X$ gives rise to a linear functional on

Y. Let $\sigma(X,Y)$ be the weakest topology on X which makes all the functionals of the form $\langle \cdot, y\rangle$ for $y \in Y$ continuous. In terms of convergence this means that $x_\alpha \longrightarrow x$ iff $\langle x_\alpha - x, y\rangle \longrightarrow 0$. This is point-wise convergence. We treat elements of X as functions on Y. In the language of semi-norms, $x_\alpha \longrightarrow x$ iff $\langle x_\alpha - x, y\rangle \longrightarrow 0$ for all $y \in Y$ iff $p_y(x_\alpha - x) \longrightarrow 0$ for all $y \in Y$, where p_y is the semi norm on X given by $p_y(x) = |\langle x, y\rangle|$. Thus $\sigma(X,Y)$ is given by a family of semi-norms so locally convex.

Since $\sigma(X,Y)$ is the topology of point-wise convergence, we can identify a base for N_0, being

$$V_{\varepsilon,y_1,\ldots,y_n} = \{x \in X : |\langle x, y\rangle| < \varepsilon \forall i = 1,\ldots,n\}.$$

Exercise 6. Conditions in the definition of dual pairs require means that the two induced topologies on X and Y are Hausdorff.

Example 18. Let X be a normed space. Consider the dual pair $\langle X, X^*\rangle$. Then $\sigma(X,X^*)$ is the weak topology on X. This is the weak* topology on X^*. Then $\sigma(X^*,X^{**})$ is also the weak* topology on X^*.

Recall, the Alaoglu-Bourbaki Theorem for Banach spaces. Let X be a Banach space. Then B_{X^*} is w^*-compact. We deduced from the Krein-Milman Theorem that $Y = C[0,1]$ over $\mathbb{R}$ then B_Y is not compact in any locally convex topology (not enough extreme points). Therefore $Y = C[0,1]$ is not equal to X^* for aany Banach space X, as otherwise $B_Y = B_{X^*}$ would be $w*$-compact. Thus $C[0,1]$ is not a dual space. If X is reflexive then $(X^*, w*) = (X^*, w)$ so that B_{X^*} is weakly compact. We will later show that this is an iff.

Definition 28. Given a dual pair $\langle X, Y\rangle$ and a subset A of X we defined the polar of A as a subset Y then

$$A^\circ = \{y \in Y : \forall x \in A : |\langle x, y\rangle| \leq 1\}.$$

Example 19. If X is a Banach space and $Y = X^*$ and $A = B_X$ then $B_X^\circ = \{f \in X^* : \forall x \in B_X : |f(x)| \leq 1\} = B_{X^*}$.

Some properties, $(\lambda A)^\circ = \frac{1}{\lambda}A^\circ$ for $\lambda > 0$. If $A \subset B$ then $B^\circ \subset A^\circ$. $(A \cup B)^\circ = A^\circ \cap B^\circ$. Note that $(A \cap B)^\circ \neq A^\circ \cup B^\circ$ always. A° is absorbing and convex and $\sigma(X,Y)-$closed. Further $A \subset A^{\circ\circ}$ (which is easy), but $A \neq A^{\circ\circ}$ in general (just take any non-convex A).

Theorem 18. *Let $\langle X, Y\rangle$ be a dual pair, $A \subset X$, be non-empty. Then $A^{\circ\circ} = \overline{\text{aconv}}^{\sigma(X,Y)} A$.*

As a corollary we have the $A = A^{\circ\circ}$ iff A is absolutely convex and σ(X, Y)-closed.

Proof. Let $C = \overline{\text{aconv}}^{\sigma(X,Y)} A$. Since $A \subset A^{\circ\circ}$ one has $A^{\circ\circ}$ is absolutely convex so $\text{aconv} A \subset A^{\circ\circ}$. $A^{\circ\circ}$ is $\sigma(X,Y)$ closed so $C \subset A^{\circ\circ}$.

We are left to prove that $A^{\circ\circ} \subset C$. Suppose not. Take $a \in A^{\circ\circ} \setminus C$. C is convex and $\sigma(X,Y)$ closed, $\{0\}$ is compact, so by seperation theorem there is $f \in (X, \sigma(X,Y))^* = Y$ such that $\sup_C f < f(a)$. Scaling f we may assume that $\sup_C f \leq 1 < f(a)$. Thus for all $x \in C$ we have $f(x) \leq 1$, but now C is balanced, so $\pm x \in C$ so $f(\pm x) = \pm f(x) \leq 1$. In particular, for all $x \in A$ we have $f(x) \leq 1$ (since $A \subset C$). But this means that $f \in A^\circ$ and since $a \in A^{\circ\circ}$ and so $|f(a)| \leq 1$ because $f(a) > 1$. □

Corollary 6. *If A and B are non-empty and absolutely convex and $\sigma(X,Y)$ closed then $(A \cap B)^\circ = \overline{\mathrm{aconv}}^{\sigma(X,Y)}(A^\circ \cup B^\circ)$.*

Proof. Bipolar Theorem. □

Theorem 19 (Aloglu-Bourbaki Theorem For Polar)**.** *Let X be a LCTVS (Hausdorff) and $V \in N_0$. Then V° with respect to the dual pair $\langle X, X^* \rangle$ is $\sigma(X, X^*)$ compact.*

Corollary 7. *The Theorem for Banach spaces is a special case. We just take $V = B_X$ and thus $V^\circ = B_{X^*}$. $\sigma(X, X^*)$ is the weak* topology on X^*.*

Proof. $V^\circ \subset X^* \subset F^X$. For $(F = \mathbb{R})$. The $\sigma(X^*, X)$ topology on X^* is just the extension of the product topology on $\mathbb{R}^X$. (That is, the topology of point-wise convergence to X^*). Call this topology τ. It suffices to prove that V° is τ-compact. We will show that it is a τ-closed subset of a τ-compact set.

Claim 1 is that V° is τ-closed in $\mathbb{R}^X$. Let (f_α) be a net in V° such that $f_\alpha \longrightarrow_\tau f$ for some $f \in \mathbb{R}^X$. Then f_α converges to f point-wise and thus f is linear. For each $x \in V$, $|f_\alpha(x)| \leq 1$ for all α because $f_\alpha \in V^\circ$. Passing to the limit in α we get $|f(x)| \leq 1$. So $f \in V^\circ$, so V° is closed. We claim now that V° is point-wise bounded. Indeed fix $x \in X$ and since V is absorbing we can find $\lambda_x > 0$ such that $x/\lambda_x \in V$. For every $f \in V^\circ$ we have $|f(x/\lambda_x)| \leq 1$, so $|f(x)| \leq \lambda_x$. This means that f is bounded. But now we view $\mathbb{R}^X = \prod_x \mathbb{R}$ and so $V^\circ \subset \prod_x [-\lambda_x, \lambda_x]$. The right hand side is $\tau - compact$ by Tychonoff theorem and so we conclude our result. □

If A is closed, then conv$A$ need not be closed. Take for example the bell curve in the plane and consider the convex hull. In a LCTVS, if A is bounded then aconv$A$ is bounded. Indeed, take any $U \in N_0$, then find absolutely convex $V \subset U$, such that $V \in N_0$. Since A is bounded, we can find $\lambda > 0$ such that $A \subset \lambda V \subset \lambda U$.

Lemma 20. *Let X be a TVS. Let $A_1, \ldots, A_n$ be aconvex compact subsets of X. Then* $\mathrm{aconv}(A_1 \cup \ldots A_n)$ *is compact.*

Proof. Take $B = \mathrm{aconv}(A_1 \cup \ldots, A_n) = \{\sum_{i=1}^n \lambda_i x_i : x_i \in A_i, \lambda_i \in F, \sum_{i=1}^n |\lambda_i| = 1\}$. Let $K = B_{\ell_1^n} \times A_1 \times \ldots \times A_n$. By Tikhomov's Theorem, K is compact. $B = f(K)$ where $f : K \longrightarrow X$ where $f(\lambda, x_1, \ldots, x_n) = \sum_{i=1}^n \lambda_i x_i$. f is continuous, so $f(K)$ is compact. □

Corollary 8. *If F is finite, then* aconv$F$ *is compact. (In a TVS).*

Theorem 21 (Mazur)**.** *Let X be a LCTVS (Hausdorff), and $A \subset X$. If A is totally bounded then $\overline{\text{aconv}}A$ is totally bounded.*

Proof. Let $V \in N_0$, be absolutely convex. Since A is totally bounded, we can find a finite set F such that $A \subset F + V$. Let $x \in \text{aconv}A$. Then $x = \sum_{i=1}^n \lambda_i]x_i$ where $x_i \in A$ and $\sum_{i=1}^n |\lambda_i| = 1$. Then $x_i \in F + V$so $x_i = f_i + v_i$ for some $f_i \in F$ and $v_i \in V$. Then $x = \sum_{i=1}^n \lambda_i(f_i + v_i) = \sum_{i=1}^n \lambda_i f_i + \sum_{i=1}^n \lambda_i v_i \in \text{aconv}F + V = \overline{\text{aconv}}F + V$. Then $a\text{conv}A \subset a\text{conv}F + V$ so taking closures we see the result.

□

Now we move onto Mackey-Arens Theory. We move onto some definitions. Let X be a TVS and $f : X \longrightarrow F$ be a linear functional. f is continuous means for all $\varepsilon > 0$ there is $V \in N_0$ such that for all $x \in V$, $|f(x)| < \varepsilon$.

Let A be a set of linear functionals osn X. We say that A is equicontinuous if for all $\varepsilon > 0$ there is $V \in N_0$ such that for all $x \in V$ and for all $f \in A$ one has $|f(x)| < \varepsilon$. This implies $A \subset X^*$. This is a uniform continuity for functions.

Lemma 22. *Let $A \subset X^*$. Then A is equicontinuous if and only if $A \subset W^\circ$ for some $W \in N_0$.*

Proof. Take $\varepsilon = 1$. Then there is $V \in N_0$ such that for all $f \in A$ and all $x \in V$ we have $|f(x)| \leq 1$, so $A \subset V^\circ$. On the contrary, if $\varepsilon > 0$ then take $V = \varepsilon W$. Then for all $f \in A$, $f(V) = \varepsilon f(W) \subset \varepsilon[-1, 1] = [-\varepsilon, \varepsilon]$. □

Recall in a LCTVS, polars are $\sigma(X^*, X)-$compact by Alaoglu-Bourbaki so every equicontinuous set is relatively $\sigma(X^*, X)-$compact.

Definition 29. Let $\langle X, Y\rangle$ be a dual pair. Think of elements of Y as functions from X to F. Let $A \subset X$. We say that a net $(y_\alpha) \subset Y$ converges to $y \in Y$ uniformly on A if for all $\varepsilon > 0$ there is α_0 such that for all $\alpha \geq \alpha_0$ and for all $x \in A$ we have $|y_\alpha(x) - y(x)| < \varepsilon$.

Again by shifting it suffices to consider convergence to zero because $y_\alpha \longrightarrow y$ iff $y_\alpha - y \longrightarrow 0$. This means $y_\alpha \longrightarrow 0$ uniformly on A iff $\sup_{x\in A} |y_\alpha(x)| \longrightarrow 0$ as $\alpha \longrightarrow \infty$. But $\rho_A(y) = \sup_{x\in A} |y_\alpha(x)|$ is almost a seminorm. From now on we will assume that A is $\sigma(X, Y)$-bounded. This guarentees that $\rho_A(y)$ is finite for all $y \in Y$, and thus $\rho_A(y)$ is finite for all $y \in Y$. Then ρ_A is a semi-norm, so uniform convergence on A is a seminorm convergence, so it corresponds to a locally convex topology (generally, not Hausdorff, i.e. the seminorms kernel is not trivial). Let B be the unit ball of ρ_A. One has $B \subset Y$. Then for all $y \in B$ one has $\rho_A(y) \leq 1$ iff $\sup_{x\in A} |y(x)| \leq 1$ iff $\forall x \in A$ one has $|y(x)| \leq 1$ iff $y \in A^\circ$.

Recall, let $\langle X, Y\rangle$ be a dual pair. Let $A \subset X$. On Y consider uniform convergence on A. We say that $y_\alpha \longrightarrow 0$ uniformly in A if for all $\varepsilon > 0$ there is a_0 such that for all $a \geq a_0$ and all $x \in A$ we have $|\langle x, y_\alpha\rangle| < \varepsilon$. Similarly for a set $B \subset Y$ we can consider uniform convergence in B as a topology on X. This convergence is given by a semi-norm $\rho_A(y) = \sup_{x\in A}\langle x, y\rangle$. The unit ball of ρ_A is A°. Multiply A° form a base of the the topology (this is generally not Hausdorff). Now let S be any family of $\sigma(X,Y)$-bounded subsets of X. Consider on Y the topology given by uniform convergence on all members of this family. This is called the S-topology). This topology is given by the family of seminorms $\{\rho_A : A \in S\}$. For a base of neighbourhoods one can take the polars of the sets in S, their scalar multiples and finite intersections.

Example 20. Let S be the set of all singletons in X. Then S convergence is point-wise convergence. $\sigma(X,Y)$ convergence.

Example 21. Let X be a Banach space. Take $S = \{B_X\}$. Then $\langle X, X^* \rangle$ S convergence is norm convergence with respect to $\|\cdot\|_{X^*}$.

We have that S topologies are locally convex, but more is true: Every locally convex topology is an S topology.

Theorem 23. *Let (X,τ) be a LCTVS. Then τ is the S topology for (X,X^*) where $X^* = (X,\tau)^*$ and $S = \{V^\circ : V \in N_0^\tau\}$. That is, $x_\alpha \longrightarrow 0$ in τ iff $x_\alpha \longrightarrow 0$ uniformly on V° for each $V \in N_0^\tau$.*

Proof. Suppose $x_\alpha \longrightarrow 0$ in τ. Fix $V \in N_0^\tau$. Take $\varepsilon > 0$ and then we have $\varepsilon V \in N_0^\tau$ so εV contains a tail of (x_α). There is α_0 such that for all $\alpha \geq \alpha_0$ we have $x_\alpha \in \varepsilon V$ so $x_\alpha/\varepsilon \in V$, for each $v \in V^\circ$ we have $|y(x_\alpha/\varepsilon)| \leq 1$ so $|y(x_\alpha)| \leq \varepsilon$.

Now suppose that $x_\alpha \longrightarrow_S 0$. Want to show that $x_\alpha \longrightarrow_\tau 0$. Take $V \in N_0^\tau$. It suffices to show that x_α has a tail in V. WLOG V is absolutely convex and τ closed since every LCTVS has a absolutely convex closed base of neighbourhoods. We have $V^\circ \in S$ so (x_α) converges to zero uniformly in V°. Take $\varepsilon = 1$. Then there is α_0 such that for all $\alpha \geq \alpha_0$ we have for all $y \in V^\circ$, $|\langle x, y \rangle| \leq 1$. Then $x_\alpha \in V^{\circ\circ} = V$ by polar theorem because V is absolutely convex and τ-closed, so weakly closed because $\sigma(X,X^*)$ and τ have the same dual X^* so the same convex closed sets. $\square$

Recall a subset A of X^* is τ-equicontinuous iff $A \subset V^\circ$ for some $V \in N_0^\tau$.

Corollary 9. *Let (X,τ) be a LCTVS. Then τ is the S topology of $\langle X, X^* \rangle$ where S is the set of all equicontinuous subsets of X^*.*

Thus every locally convex topology is an S topology. The new idea is to relate properties of τ with properties of S. The idea is to realate properties of τ with properties of S. Recall, given a dual pair $\langle X, Y \rangle$ and a locally convex topology τ on X we say that τ is compatible with $\langle X, Y \rangle$ if $(X,\tau)^* = Y$. We already know that $\sigma(X,Y)$ is countable. Know that all compatible topologies have the same closed sets.

Theorem 24 (Mackey Arens)**.** *Let $\langle X, Y \rangle$ be a dual pair and τ a locally convex Hasudorff topology on X. τ is compatible with $\langle X, Y \rangle$ iff τ is the S topology for some S consisting of absolutely convex $\sigma(Y,X)$-compact sets. Also $\bigcup S = Y$.*

Proof. Suppose that τ is compatible, $(X, \tau)^* = Y$. By the previous theorem, τ is the S topology where S is the set of all polars $S = \{V^\circ : V \in N_0^\tau\}$, where each polar is absolutely convex and by Banach-Alaoglu, V° is $\sigma(X^*, X)$-compact. To show that $\bigcup S = Y$ take $y \in Y = X^*$ y is a $\tau-$continuous functional on X, so there is $V \in N_0$ such that $y(V) \subset [-1, 1]$. But thus $y \in V^\circ$, so $y \in \bigcup S$ because $V^\circ \in S$. The reverse proof we will skip. $\square$

Theorem 25 (Mackey)**.** *All compatible topologies have the same bounded sets.*

Let $\langle X, Y\rangle$ be a dual pair. We know that $\sigma(X, Y)$ is the weakest compatible topology on X. Let S be the set of all absolutelt convex $\sigma(Y, X)$ compact subsets of Y. Then the corresponding S topology on X is the strongest topology compatible topology on X. Denoted by $\tau(X, Y)$, the Mackey topology. A topology τ on X is compatible iff $\sigma(X, Y) \leq \tau \leq \tau(X, Y)$. This is NOT the strong topology $\beta(X, Y)$.

Recall that the canonical inclusion $\iota : X \longrightarrow X^{**}$ is a homeomorphism $(X, w) \longrightarrow_\iota (X^{**}, w^*)$. Then we have Goldstein's Theorem: If X is a normed space then $\iota(B_X)$ is w^*-dense in $B_{X^{**}}$, so $\iota(X)$ is w^*-dense in X^{**}.

Corollary 10. *X is reflexive iff B_X is $w-$compact.*

Proof. For the forward, this means ι is onto so $B_{X^{**}} = \iota(B_X)$. Then $(B_X, w) \cong (B_{X^{**}}, w^*)$, which is compact by Banach Alogolu Theorem.

For the reverse, if (B_X, w) is compact, then $\iota(B_X)$ is w^*-compact so w^*-closed. By Goldstein's Theorem, it is also w^*-dense in $B_{X^{**}}$, so this means that $\iota(B_X) = B_{X^{**}}$. Then $\iota(X) = X^{**}$ by scaling, so ι is onto, so X is reflexive. $\square$

Note that if one has a topological space which is metrizable, then it must be first countable, so there is a countable base of neighbourhoods at each point. If (X, w) is the weak topology, we have a base of zero neighbourhoods of the form $V_{\varepsilon, f_1, \ldots, f_n} = \{x \in X : |f_i| < \varepsilon \,\forall i = 1, \ldots, n\}$. WLOG we can take $\varepsilon = 1$ and scale the f_i accordingly. Thus, we will use the notation $V_F = V_{1, f_1, \ldots, f_n}$ to simplify notation. Similalry, for (X^*, w^*) we have base neighbourhoods $V_A = V_{1, x_1, \ldots, x_n}$. Today we ask the question, if (X, w) or (X^*, w^*) are metrizable? The answer is NOOO!

Lemma 26. *TFAE:* $\dim X < \infty$*,* (X, w) *is metrizable,* (X, w^*) *is metrizable.*

Proof. We prove only $1 \iff 2$, 3 is easy. If $\dim X < \infty$ then all linear topologies are the same, so since X is normable, every linear topology is metrizable. On the other hand, if we go by contradiction and suppose that (X, w) is metrizable but $\dim X = \infty$ then (X, w) is first countable, so there exists a sequence (F_n) of finite subsets of X^* such that V_{F_n}'s form a base of zero for (X, w). Put $F = \bigcup_n F_n$, a countable set in X^*. Let $Y = \text{span}F$. We claim that $Y = X^*$. Clearly $Y \subset X^*$. If not, there is $g \in X^* \setminus Y$, $g \neq 0$. Then $V_{1,g}$ is in N_0^w. Then $V_{F_n} \subset V_{\varepsilon, g}$ for some n. If $x \in \bigcap_{f \in F_n} \ker g$ then $\lambda x \in \bigcap_{f \in F_n} \ker f$ for every $\lambda \in F$, so $\lambda x \in V_{F_n}$, so $\lambda x \in V_{1,g}$, so $|g(\lambda x)| \leq 1$, since λ is arbitary, then $g(x) = 0$, so

$x \in \ker g$. Thus $\bigcap_{f\in F_n} \ker f \subset \ker g$, but then this means $g \in \text{span} F_n$, so $g \in Y$. This proves the claim, so $X^* = Y = \text{span} \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \text{span} F_n$. Thus X^* can be written as a countable union of finite dimensional subspaces which are closed and nowhere dense, which contradicts Baire-Category theorem. $\square$

Now we talk about the balls

Lemma 27. *X is seperable iff (B_{X^*}, w^*) is metrizable.*

Proof. If X is seperable, then we can find a sequence $\{x_n\} \subset X$ which is dense in X. For a net f_α and f in B_{X^*}, by definition this means $f_\alpha \longrightarrow_{w^*} f$ iff for all $x \in X$ one has $f_\alpha(x) \longrightarrow_\alpha f(x)$ iff for all n $f_\alpha(x_n) \longrightarrow_\alpha f(x_n s)$. The reason why this works is because we have a bounded net of functions. But this holds iff $\rho_{x_n}(f_\alpha - f) \longrightarrow 0$ where $\rho_{x_n}(f) = |f(x_n)|$, this means that the w^*-topology is given by sequence of semi-norms, where we define

$$d(f,g) = \sum \frac{\rho_n(f-g)}{1+\rho_n(f-g)} 2^{-n},$$

a metric on B_{X^*} corresponding to w^*.

For the other direction, suppose that (B_{X^*}, w^*) is metrizable, so it is first countable, so there is a countable base at zero. We can find a sequence $\{A_n\}$ of finite subsets of X such that $(V_{A_n} \cap B_{X^*})$ form a base of neighbourhoods for (B_{X^*}, w^*). But now, let $A = \bigcup_{n=1}^{\infty} A_n$, countable. It suffices to show that $X = \overline{\text{span}} A$. Suppose not, then $\overline{\text{span}}$ is a proper closed subspace, so there is $f \in X^*$, $f \neq 0$, f vanishes on $\overline{\text{span}}(A)$. Hence f vanishes on each A_n, so $f \in V_{A_n}$, so $f \in \bigcap_{n=1}^{\infty} V_{A_n} = \{0\}$, so $f = 0$, a contradiction. $\square$

Corollary 11. *If X is seperable then the restriction of w^* to every bounded subset of X^* is metrizable.*

Now we give the dual version.

Lemma 28. *X^* is seperable iff (B_X, w) is metrizable.*

Proof. Suppose that X^* is seperable, then (B_{X^**,w^*}) is metrizable, let d_0 be a metric. Then $(B_X, w) \longrightarrow (B_{X^{**},w*})$ is a homeomorphic embedding. For $x, y \in B_X$ we define $d(x,y) = d_0(\iota(x), \iota(y))$. $d(x_n, x) \longrightarrow 0$ iff $d_0(\iota(x_n), \iota(x)) \longrightarrow 0$ iff $\iota(x_n) \longrightarrow_{w^*} \iota(x)$ iff $x_n \longrightarrow_w x$.

On the other hand, suppose that (B_X, w) is metrizable, so first countable. Find a sequence (F_n) of finite subsets of X^* such that the sequence $(V_{F_n} \cap B_X)$ is a base of zero for (B_X, w). Put $F = \bigcup_{n=1}^{\infty} F_n$. Suffices to prove that $X^* = \overline{\text{span}} F$. Suppose not, then there is $g \in X^* \setminus \overline{\text{span}} F$, $g \neq 0$. WLOG $\|g\| = 1$. Now by Hahn Banach theorem we can find $\xi \in X^{**}$ such that ξ vanishes on $\overline{\text{span}} F$, $\xi(g) \neq 0$, $\|\xi\| = 1$, so $\xi \in B_{X^{**}}$. By Goldstein's theorem, we can find a net $x_\alpha \subset B_X$ such that $\iota(x_\alpha) \longrightarrow_{w^*} \xi$. Thus for $\varepsilon > 0$ we know $V_{\varepsilon,g} \cap B_X$ is a neighbourhood of 0 (B_X, w). This must contain a base neighbourhood $V_{F_n} \cap B_X$

for some n, so $V_{F_n} \cap B_X \subset V_{\varepsilon,g} \cap B_X$. For each $f \in F_n$ it follows from $\hat{x}_\alpha \longrightarrow_w \xi$ that $f(x_\alpha) \longrightarrow \xi(f) = 0$. This implies $f(x_\alpha) \longrightarrow 0$ for each $f \in F_n$. Then there is α_0 such that for all $\alpha \geq \alpha_0$ and all $f \in F_n$ one has $|f(x_\alpha)| < \varepsilon$, because F_n is finite. This means $x_\alpha \in V_F$. So $x_\alpha \in V_{\varepsilon,g}$, but this means $|g(x_\alpha)| < \varepsilon$. Hence, $g(x_\alpha) \longrightarrow 0$. But $\hat{x}_\alpha \longrightarrow_{w^*} \xi$ so $g(x_\alpha) = \hat{x}_\alpha(g) \longrightarrow \xi(g) \neq 0$. □

Theorem 29 (Eberlein-Smulyan's Theorem)**.** *Weak compactness is sequential.*

In general, weak topology is not 1st countable, hence not sequential, so we need to use nets. A set A is a Hausdorff space is said to be relatively compact if cl$A$ is compact, or equivalently, if A is contained in a compact set, or equivalently every net in A has an accumulation point, or equivalently every net has a convergent subset whose limit is not necessarily in A. (In particular, every sequence in A has a convergent subnet). In the first countable case, it suffices to show that every sequence has a convergent subsequence. Recall that if X is a seperable Banach space and $C \subset X$ is $w-$compact then the weak topology in C is metrizable, hence first countable.

Observe that every weakly compact set is bounded. Weakly compact implies weakly bounded which is the same as norm bounded in a Banach space. This immediately means also that relatively weakly compact sets are bounded by the same type of argument. If X is reflexive then relatively weakly compact sets are exactly the bounded sets. The forward we already know, and the other way is because the B_X is weakly compact.

If A is bounded thne $A \subset \lambda B_X$ for some $\lambda > 0$ λB_X is weakly compact, so A is relatively weakly compact.

Lemma 30. *Let $A \subset X$ where X is a Banach space. Then A is relatively weakly compact iff A is bounded and $\overline{(\iota(A))}^{w^*} \subset \iota(X)$.*

From here on out we drop the notation $\iota(x)$, and just write x. In the above notation we simply mean that $\overline{A}^{w^*} \subset X$.

Proof. $\Longrightarrow$. Suppose A is relatively weakly compact. This means that $\overline{A}^{w}$ is weakly compact in X, so in particular bounded. Let $\xi \in \overline{A}^{w^*}$ (in X^{**}). Then we can find a net (x_α) in A such that $(x_\alpha) \longrightarrow_{w^*} \xi$ in X^{**}. Since A is relatively weakly compact then (x_α) has a weakly convergent subnet, we have $(x_\alpha) \longrightarrow_w x \in X$. This means that $x_\alpha \longrightarrow_{w^*} x \in X \subset X^{**}$. This means $\xi \in X$. $\Longleftarrow$. Suppose that $\overline{A}^{w^*} \subset X$ in X^{**} and A is bounded. Without loss of generality, $A \subset B_X$, so $A \subset B_{X^{**}}$ in X^{**}. But $B_{X^{**}}$ is w^*-compact by Alogolu-Bourbaki, so A is relatively-w^* compact, i.e. clA^{w^*} is w^*-compact, hence this cl$A$ is weakly compact as a subset of X, so A is relatively weakly compact. □

Theorem 31 (Restatement of Eberlein Smulyan.)**.** *Let $A \subset X$ where X is a Banach space. The following are equivalent: A is relatively weakly compact.*

Every sequence in A has a $w-$convergent subsequence. Every countable subset of A has a weak accumulation point.

Proof of Eberlein-Smulyan. $\implies$ Suppose A is relatively weakly compact. Let (a_n) in A, and put $C = \overline{A}^w$. Then C is weakly compact. Put $Y = [a_n]$ so Y is seperable. Being a closed subspace, Y is weakly closed, so $C \cap Y$ is weakly closed subset of C, hence weakly compact in X and therefore in Y. Since Y is seperable, $(C \cap Y, w)$ is metrizable, so first countable, and $(a_n) \subset C \cap Y$, so we can find a convergent subsequence.

2 $\implies$ 3 is trivial.

3 $\implies$ 1. Suppose A satisfies 3. We claim that A is bounded. For each $f \in X^*$, $f(A)$ satisfies 3 in F. Every countable subset of $f(A)$ has an accumulation point. Hence, $f(A)$ is relatively compact in F, so bounded. This means that A iw weakly bounded since this is true for all $f \in X^*$. But then by uniform boundedness principle, norm bounded. By the Lemma above, it suffices to prove that $\overline{A}^{w^*} \subset X$ in X^{**}. Let $\xi \in \overline{A}^{w^*}$, we will construct a sequence (a_n) in A. By 3, (a_n) has an accumulation point $a \in X$. We will show that $\xi = a$ (or rather $\xi = \hat{a}$), so $\xi \in X$. We will inductively construct three sequences, one sequence $(a_n) \in A$, $(f_n) \subset S_{X^*}$, a sequence (k_n) in $\mathbb{N}$ which is strictly increasing such that, for all $\zeta \in \operatorname{span}\{\xi, a_1, \ldots, a_n\}$, then there exists $i \leq k_{n+1}$ so that $\langle \zeta, f_i \rangle \geq (1/2) \|\zeta\|$. That is, $f_1, \ldots, f_{k_n}, f_{k_{n+1}}$ almost norm the subspace. The next property is that $|\langle \xi - a_n, f_i \rangle| < 1/n$ for all $i = 1, \ldots, k_n$. That is $a_n \approx \xi$ when witnessed by $f_1, \ldots, f_{k_n}$.

Lets start by considering $n = 1$. Take $f_1 \in S_{X^*}$. Put $k_1 = 1$. Since $\xi \in \overline{A}^{w^*}$, every w^* neighbourhood of ξ meets A, so $\xi + V_{1,f_1}$ meets A. Take any $a \in (\xi + V_{1,f_1}) \cap A$. Then $a_1 \in A$ and $|\langle a_1 - \xi, f_1 \rangle| < 1$, so we get 2. Suppose now we constructed $a_1, \ldots, a_n$, and $k_1, \ldots, k_n$ and $f_1, \ldots, f_n$. Put $M = \operatorname{span}\{\xi, a_1, \ldots, a_n\}$ in X^{**}. Then S_M is compact, so we can find $\zeta_{k_n+1}, \ldots \zeta_{k_{n+1}}$ such that for all $\zeta \in S_M$ one has $\|\zeta - \zeta_i\| \leq 1/4$. For each $i = k_n+1 \ldots, k_{n+1}$, find $f_i \in S_{X^*}$ such that $\langle f_i, \zeta_i \rangle > 3/4$. For each $\zeta \in S_M$ find $i = k_{n+1}, \ldots k_{n+1}$ with $\|\zeta - \zeta_i\| < 1/4$. This means $\langle f_1, \zeta \rangle = \langle f_1, \zeta - \xi_i + \xi_i \rangle = \langle f_i, \zeta - \xi_1 \rangle + \langle f_i, \xi_i, > \rangle\, 1/2$. Now we want to satisfy $|\langle \xi - a_n, f_i \rangle| < 1/n$ for all $i = 1, \ldots, k_n$. Since $\xi \in \overline{A}^{w^*}$, A meets $\xi + V_{1/(n+1), f_1, \ldots, f_{k_{n+1}}}$. Take any $a_{n+1} \in A \cap (\xi + V_{1/(n+1), f_1, \ldots, f_{k_{n+1}}})$. Then $a_{n+1} \in A$ and further $|\langle \xi - a_{n+1}, f_i \rangle| < 1/(n+1)$ for all $i = 1, \ldots, k_{n+1}$. By assumption, (a_n) has a weak accumulation point $a \in X$. We show now $\xi = a$, let $Y = \operatorname{span}\{a_1, a_2, \ldots, a_3\}$. $Y \subset X$, and $a \in \overline{Y}^w = \overline{Y}$. Then if we consider $Y \subset B = \{\xi, a_1, a_2, \ldots\} \subset X^{**}$, so $\overline{Y} \subset \overline{B}$, so $a \in \overline{B}$. Also, $\xi \in \overline{B}$. So $\xi - a \in \overline{B}$. We can find $\zeta \in B$ such that $\|\zeta - (\xi - a)\| \leq (1/4) \|\xi - a\|$, WLOG $\|\zeta\| = \|\xi - a\|$. Since $\zeta \in B$, $\zeta \in \operatorname{span}\{\xi, a_1, \ldots, a_n\}$ for some n, by (a), find $j \leq k_{n+1}$ such that $\langle f_j, \zeta \rangle \geq (1/2) \|\zeta\| = (1/2) \|\xi - a\|$. Then we can write

$$\langle f_j, \zeta \rangle = \langle f_j, \zeta - (\xi - a) \rangle + \langle f_j, \xi - a \rangle \leq (1/4) \|\xi - a\| + \langle f_j, \xi - a \rangle,$$

so $\langle f_j, \xi - a \rangle \geq (1/4) \|\xi - a\|$. Let $\varepsilon > 0$, since a is a weak accumulation point of (a_n) we can find infinitely many a_m's in $a + V_{\varepsilon, f_j}$. This precisely means that $|\langle f_j, a - a_m \rangle| < \varepsilon$. When $m > j$, then we have $|\langle f_j, \xi - a \rangle| \leq |\langle f_i, \xi - a_m \rangle| + |\langle f_j, a - a_m \rangle| \leq 1/m + \varepsilon$. Since this is true for infinitely many m, $|\langle f_j, \xi - a \rangle| \leq \varepsilon$. By the earlier remark, $(1/4)\, \|\xi - a\| < \varepsilon$. Since ε is arbitary, $\xi = a$. □

Corollary 12. *A is weakly compact iff every seqeuence in A has a subsequence which convergences to an element in A.*

Proof. If A is weakly compact, then it is definitely relatively weakly compact, so by $E - S$ theorem (the above), every sequence in A has a weakly convergent subsequence in X, but since A is weakly compact, it is weakly closed, and thus this subsequence has limit in A.

For the reverse, use $E - S$ theorem to prove that A is relatively weakly compact. It is left to show that it is weakly closed. Take $\xi \in \overline{A}^{w^*}$, argue as in the proof above that ξ is the weak limit of a sequence in A (write it down carefully). □

Corollary 13. *A is relatively weakly compact iff every countable subset of it is.*

Proof. The forward is trivial. For the reverse, every countable subset of A is relatively weakly compact, hence it is a weak accumulation point, so by $E - S$ theorem, A is relatively weakly compact. □

This is useful because we can always assume WLOG that A is countable to prove that it is.

Corollary 14. *X is reflexive iff every bounded sequence has a weakly convergent subsequence.*

Proof. X is reflexive iff B_X is weakly compact iff (by $E - S$) every sequence in B_X has a weakly convergent subsequence. □

Recall from homework, B_X is always weakly closed, because if $x_\alpha \longrightarrow_w x$ then $\|x\| \leq \liminf \|x_\alpha\|$. This implies B_X is weakly compact iff B_X is relatively weakly compact. MIDTERM CUTOFF

Now we state the Dunford-Pettis theorem (without proof). We know that relatively weakly compact implies bounded. For reflexive spaces relatively weakly compact sets are precisely the bounded ones. How do bounded sets look like in non-reflexive spaces, say $L_1(\mu)$? Here is an example of a bounded set in $L_1(\mu)$ that is not relatively weakly compact. Let (A_n) be a disjoint sequence of measurable sets, $\mu(A_n) \longrightarrow 0$. Denote $f_n =$ the normalized characterized functions of A_n. $f_n = \frac{1}{\mu(A_n)}\chi_{A_n}$. Definitely this sequence is bounded because $\|f_n\| = 1$. (f_n) does not converge weakly though. Let $g = 1$ on A_{2n} and -1 on A_{2n-1} and 0 elsewhere. Then $g \in L_\infty$, $\|g\| = 1$. Then $\langle g, f_{2n}\rangle = 1$, $\langle g, f_{2n-1}\rangle = -1$, so $\langle g, f_n\rangle$ does not converge, so (f_n) does not converge weakly. Let $F = \{f_n\}$. Similarly, no subsequence of (f_n) is weakly compact. This proves that F is not realtively weakly compact by $E - S$ theorem. F is bounded but not relatively weakly compact. Recall from measure theory, if $f \in L_1(\mu)$ then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ then $\int_A |f| < \varepsilon$. On the other hand this property fails for a family of functions (example above).

Definition 30. A set of functions F is uniformly integrable (or equi-integrable) if the above condition is true, but uniformly. That is, $\varepsilon > 0$ there is $\delta > 0$ such that for all $f \in F$ one has $\int_A |f| < \varepsilon$.

Definition 31. For $f, g \in L_1(\mu)$, $f \leq g$ if $f(t) \leq g(t)$ a.e.

We define an order interval of $L_1(\mu)$ as $[f, g] = \{h \in L_1(\mu) : f \leq h \leq g\}$, for $f \leq g$. We say that a subset F of $L_1(\mu)$ is order bounded if it is contained in an order interval $\iff \exists h \geq 0 : F \subset [-h, h]$. F is almost order bounded if $\forall \varepsilon > 0$ there is $h \geq 0$ such that $F \subset [-h, h] + \varepsilon B_X$.

Define an operator $T : \ell_1 \longrightarrow L_1[0,1]$ via $Te_n = f_n$. Then T is an isometric embedding.

$$\left\| T \sum(\alpha_i e_i) \right\| = \left\| \sum(\alpha_i f_i) \right\| = \sum |\alpha_i| = \left\| \sum(\alpha_i e_i) \right\|.$$

We call the embedding the isomorphic copy of ℓ_1.

Theorem 32 (Dunford-Pettis Theorem). *Let F be a bounded subset of $L_1(\mu)$, for μ finite. The following are equivalent, A is relatively weakly compact, and A is uniformly integrable, F is almost order bounded. F contains no isomorphic copy of the unit vector basis of ℓ_1 (algebraic). For every disjoint sequence (A_n) of measurable sets, $\int_{A_n} |f| \longrightarrow 0$ in n uniformly on $f \in F$, For every disjoint bounded positive sequence (g_n) in $L_\infty(\mu) = L_1(\mu)^*$, $\langle g_n, f\rangle \longrightarrow 0$ uniformly on $f \in F$.*

We remark that the fifth condition is a special case of the sixth by taking $g_n = \chi_{A_n}$. The DP theorem remains valid for infinite measures, but (2) has to be adjusted.

Definition 32. Recall, $T \in L(X, Y)$ is compact if TB_X is relatively compact in Y. Equivalently, for every bounded sequence (x_n) in X, the sequence (Tx_n) has a convergence subsequence.

Lemma 33. *If T is compact and $x_n \longrightarrow_w x \in X$, then $Tx_n \longrightarrow_{\|\cdot\|} x$.*

Proof. Suppose that T is compact, $x_n \longrightarrow_w 0$, then $Tx_n \longrightarrow_w 0$. Since (Tx_n) has a norm convergent subsequence, $Tx_{n_k} \longrightarrow y$, but $Tx_{n_k} \longrightarrow_w y$, so $y = 0$ by uniqueness of limits. Hence $Tx_{n_k} \longrightarrow 0$. Similarly, for every subsequence (x_{n_m}) of (x_n) there is a further subsequence $(a) = (x_{n_{m_k}})$ such that $Ta \longrightarrow 0$. Then $Tx_n \longrightarrow 0$. $\square$

Recall, that we are using $x_n \longrightarrow a \iff \forall n_k, \exists n_{k_m}$ such that $x_{k_n} \longrightarrow a$. Further suppose that X is reflexive. If $x_n \longrightarrow_w 0$, then $Tx_n \longrightarrow 0$. Let (x_n) be a bounded sequence in X. By $E-S$ theorem, (x_n) has a weakly convergent subsequence $x_{n_k} \longrightarrow_w x$. By assumption $Tx_{n_k} \longrightarrow Tx$, so T is compact. These operators without the property that X is reflexive are called Dunford-Pettis operators.

Definition 33. If $T \in L(X, Y)$, we say that T is weakly compact if TB_X is relatively compact in Y.

It follows immediately from $E-S$ theorem that T is weakly compact iff for every bounded sequence (x_n), (Tx_n) has a weakly convergent subsequence.

If Y is reflexive, then every operator $T \in L(X, Y)$ is weakly compact because TB_X is bounded, hence relatively weakly compact. If X is reflexive, then every $T \in L(X, Y)$. For each bounded sequence (x_n) in X, there is a weakly convergent subsequence, $x_{n_k} \longrightarrow_w x$, then $Tx_n \longrightarrow_w x$. Given

$$X \longrightarrow_T Y \longrightarrow_S Z,$$

we see that ST is weakly compact if S or T is.

Lemma 34. *If T factors through a reflexive Banach space, then T is weakly compact.*

In $L(X)$, weakly compact operators form a two-sided ideal.

Example 22. Consider $\iota : \ell_1 \longrightarrow \ell_\infty$, the inclusion map. ι is bounded, but ι is not compact because (ιe_n) has no convergent subsequences. However ι is weakly compact, since we can write $\ell_1 \longrightarrow \ell_2 \longrightarrow \ell_\infty$, and ℓ_2 is reflexive

Recall, X is a Banach space. A sequence $\{e_k\}$ in X is called a Schauder basis if every vector $x \in X$ admits a unique expansion $x = \sum_{k=1}^\infty a_k e_k$. The n-th basis projection $P_n(\sum_{k=1}^\infty a_k e_k) = \sum_{k=1}^n a_k e_k$. $P_n : X \longrightarrow X$ with the range of P_n equal to $\operatorname{span}\{e_1, \ldots, e_n\}$. $P_n^2 = P_n$. ALl $P_n's$ are bounded, even, uniformyl bounded : $K = \sup \|P_n\| < \infty$. K is called the basis constant of $\{e_k\}$. If $n \leq m$, $\alpha_1, \ldots, \alpha_m$ one has

$$\left\| \sum_{k=1}^n a_k e_k \right\| \leq K \left\| \sum_{k=1}^m a_k e_k \right\|.$$

$K = 1$ iff $\forall n \le m$ one has $\|\sum_{k=1}^{n} a_k e_k\| \le \|\sum_{k=1}^{m} a_k e_k\|$. Note that $K \ge 1$ because projections always have norm bigger than 1. In this case we say that (e_k) is a monotone basis. The unit vector basis of c_0 and ℓ_p $1 \le p < \infty$ are monotone. It follows from the proof thaof the theorem last time that given any Schauder basis $\{e_k\}$ we can renorm the space $\|x\| = \sup_n \|P_n x\|$, so that e_k is monotone in $(X, \|\cdot\|)$. Let $\{e_k\}$ be a basis. For each k define $e_k^* : X \longrightarrow F$ via $e_k^*(\sum_{k=1}^{\infty} \alpha_k e_k) = \alpha_k$. e_k^* is linear functional (easy). Furthermore

$$\alpha_n e_n = P_n x - P_{n-1} x$$

which implies

$$\|\alpha_n e_n\| \le \|P_n x\| + \|P_{n-1} x\| \le K\|x\| + K\|x\| = 2K\|x\|.$$

But on the other hand $\|\alpha_n e_n\| = |\alpha_n| \|e_n\| = |e_n^*(x)| \|e_n\|$. Further $|e_n^*(x)| \le \frac{2k\|x\|}{\|e_n\|}$ so e_n^* is bounded.

Example 23. Let $X = c_0$ or ℓ_p $1 < p < \infty$. Then $X^* = \ell_1$ (resp ℓ_{p^*}). Then (e_k^*) is the standard unit basis of X^*, in particular it is again a basis. On the other hand if $X = \ell_1$ then $X^* = \ell_\infty$. But (e_k^*) is not a basis of X^* because X^* is not seperable.

We know that if (e_k) is a basis, then (e_k) is linearly independent and $[e_k] = X$. The converse is false. For an example let $X = \ell_p$ $(1 \le p < \infty)$ or c_0. Let $(e_k)_{k=1}^{\infty}$ be the unit vector basis. Let $e_0 \in X$ be any vector with all non-zero coordinates, say $e_0 = (1/2, 1/4, 1/8, \ldots)$. Consider $(e_k)_{k=0}^{\infty}$. Then $[e_k]_{k=0}^{\infty} = X$ and $(e_k)_{k=0}^{\infty}$ is linearly independent, but (e_k) is not a basis because we have two different expansions. What is the proper way to go back?

Theorem 35. *Let (e_k) be a sequence in X. The following are equivalent (1): (e_k) is a basis. (2) : $e_k \neq 0$ for every k, $[e_k] = X$ and there exists $K \ge 1$ such that $\forall n \le m$, $\alpha_1, \ldots, \alpha_m$*

$$\left\|\sum_{k=1}^{n} \alpha_k e_k\right\| \le K \left\|\sum_{k=1}^{m} \alpha_k e_k\right\|.$$

Proof. $(i) \implies (ii)$ we already did. $(ii) \implies i$, we want to verify that (e_k) is linearly independent: Suppose that $\sum_{k=1}^{m} \alpha_k e_k = 0$. Apply the basis inequality with $n = 1$. Then $\|\alpha_1 e_1\| \le K \|\sum_{k=1}^{m} \alpha_k e_k\| = 0$, so $\alpha_1 = 0$. Similarly, $\alpha_2 \ldots, \alpha_m = 0$. Put $Y = \text{span}\{e_k\}$, $\bar{Y} = [e_k] = X$, Y is dense in X. For each n define $P_n : Y \longrightarrow \text{span}(e_1, \ldots, e_m)$ by $P_n(\sum_{k=1}^{m} \alpha_k e_k) = \sum_{k=1}^{\min m,n} \alpha_k e_k$ then P_n is bounded and linear. Linear is easy, but bounded because if $x \in Y$ then $x = \sum_{k=1}^{m} \alpha_k e_k$. If $n \ge m$ then $P_n x = x$ so $\|P_n x\| = \|x\|$. If $n \le m$ then $\|P_n x\| \le K\|x\|$ so $\|P_n\| \le K$. P_n extends to a bounded opeartor on X, which we will still denote by P_n. Then $\|P_n\| \le K$ and $P_n : X \longrightarrow \text{span}(e_1, \ldots, e_n)$. Then $P_n P_m = P_{\min m,n} = P_m P_n$ on Y, hence, by the uniqueness of the extension on X, also on X. Fix $x \in X$, $n \in \mathbb{N}$. $P_n x \in \text{span} e_1, \ldots, e_{n+1}$, $P_{n+1} x = \sum_{k=1}^{n+1} \alpha_k e_k$.

Then $P_n x = P_n P_{n+1} x = P_n(\sum_{k=1}^{n+1} \alpha_k e_k) = \sum_{k=1}^{n} \alpha_k e_k$. For each n $P_n x$ is the initial segemnt of $P_{n+1}x$. This means that there exists an infinite sequence $(a_k)_{k=1}^{\infty}$ such that $P_n x 2e = \sum_{k=1}^{\infty} \alpha_k e_k$. □

Recall from last time: a seuqence (e_k) in a Banach space X is a basis iff $e_k \neq 0$ for all k and $[x] = X$ and there is $K \geq 1$ such that for all $n \leq m$, $\alpha_1, \ldots, \alpha_m$:

$$\left\| \sum_{k=1}^{\infty} a_k e_k \right\| \leq K \left\| \sum_{k=1}^{m} \alpha_k e_k \right\|.$$

The least such K is the basis constant of (e_k).

Example 24. We have a Schauder basis for $C[0,1]$. Indeed let (a_k) be the dyadic sequence $(1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, 3/16, \ldots)$. This is called the Schauder basis

$$\{f_0, f_1, \ldots, f_n\},$$

(see Lecture pictures) spans all piece-wise affine functions with nodes at $d_1, \ldots, d_n$. This has basis constant 1. They are also linearly independent.

Example 25. Consider $C[0,1]$. Let $f_n(t) = t^n$. Then $\text{span}(f_n) =$ all polynomials. By Stone Weierstrass, $[f_n] = C[0,1]$. This is not a basis. If $h \in C[0,1]$. If it were a basis then $h(t) = \sum_{k=0}^{\infty} a_k t^k$, uniformly convergent series which by calculus is differentiable.

Example 26. Consider the Haar basis in $L_p[0,1]$ for $1 \leq p < \infty$. Recall the Rademacher sequence (r_n). Define $h_0 = r_0$, $h_1 = r_1$, $h_2 =$ first half of r_2 and h_3 is second half. Then $h_2 + h_3 = r_2$. Then split r_3 into four pieces and define h_4, h_5, h_6, h_7 in this way. Then $h_k = f_k'$, up to a scalar multiple where (f_k) is THE Schauder basis. If $n < m$ then either their supports are disjoint, so $h_n h_m = 0$, or $h_n h_m = \pm h_n$. In either case,

$$\int h_n h_m = 0.$$

Thus in $L_2[0,1]$ (h_k) is an orthogonal sequence. (h_k) satisfies the basis inequality with $K = 1$ because $t \longrightarrow t^p$ is a convex function. It is left to show that $[h_k] = X$. Indeed, note that all dyatic intervals are in the span (characteristic functions). For example, $\frac{1}{2}(h_0 + h_1) = \chi_{[0,1/2]}$. All simple functions are of the form a finite sum with dyadic intervals are in $[h_k]$ and as an exercise thse simple functions are dense in all simple functions, but simple functions are dense in L_p.

Example 27. What is P_n for the Haar basis? For $f \in L_p[0,1]$, $P_n f = E(f, F_n)$ where F_n is the n-th dyadic σ-algebra, the σ-algebra generated by $h_0, \ldots, h_n$.

A basis is defined in topological terms, so it is an isomorphic concept, meaning that if you renorm the space, then a basis stays a basis (but the basis constant may change). If (x_n) is a basis in X and $T : X \longrightarrow Y$ a surjective isomorphism, then (Tx_n) is a basis in Y. Does every seperable Banach space have a basis? Enfo: NO.

Definition 34. A sequence (x_n) in X is a basic sequence if it is a basis of a closed subspace. Then the subspace has to be $[x_n]$.

A non-zero sequence (x_n) is basic iff there exists $K \geq 1$ such that for all $n > m$ and $\alpha_1, \ldots, \alpha_m$ one has

$$\left\| \sum_{k=1}^{n} \alpha_k x_k \right\| \leq K \left\| \sum_{k=1}^{m} \alpha_k x_k \right\| .$$

Some examples, if $X = \ell_p$ for $1 \leq p < \infty$ or c_0 and $x_n = e_{2n}$ then (x_n) is a basic sequence, (x_n) is a basis for the even components. Mor egenerally, every subsequence of a basis is a basic sequence. Even more generally, every usbequence of a basic sequence is basic.

Example 28. Take $X = \ell_\infty$, (e_n) to be the unit vectors in ℓ_∞. Then $[e_n] = c_0$ (e_n) is a basis of c_0, so (e_n) is a basic sequence in ℓ_∞ (but not a basis).

Example 29. Take H to be a Hilbert space (x_k) and orthogonal non-zero sequence. Then (x_k) is a basis of $[x_k]$ so (x_k) is a basic sequence in H.

Example 30. Let (x_n) be a sequence in X. A block sequence is a sequence split up into blocks of any finite size, for example $y_1 = 3x_1 + 3x_2 + 7x_3$, $y_2 = 10x_4 + x_5$, $y_3 = -x_6$.

More formally, given a sequence (x_n). An increasing sequence (n_k) in $\mathbb{N}$ and an sequence (α_n) of scalars, put $y_k = \sum_{i=n_k}^{n_{k+1}-1} \alpha_i x_i$. Then (y_k) is a block sequence.

A block sequence of a basic sequence is again a basic sequence (hint, it satisfies the basis inequality).

Let X and Y be two Banach spaces. Fix a basic sequence (x_n) in X and $T : [x_n] \longrightarrow Y$ an isometric isomorphism. If we think of $T : [x_n] \longrightarrow \mathrm{im}(T) = [Tx_n]$ then we have a surjective isomoprhism, so (Tx_n) is a basis of Range T, hence a basic sequence. If $y_n = Tx_n$ we say that $(y_n) \sim (x_n)$.

Lemma 36. *Let (f_n) be a normalized disjoint sequence in $L_p(\mu)$ $1 \leq p < \infty$. Define $T : \ell_p \longrightarrow L_p(\mu)$ by $T(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_k f_k$.*

Define $Te_k = f_k$ and extend T to c_{00} by linearity. Then

$$\left\| T(\sum_{k=1}^{n} \alpha_k e_k) \right\|^p = \left\| \sum \alpha_k f_k \right\|^p = \sum |\alpha_k|^p .$$

So T is an isometry, so T extends to an isometry for ℓ_p to $L_p(\mu)$ and $T(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_k f_k$. This implies that (f_n) is a basic sequence in $L_p(\mu)$ and is equivalent to the unit vector basis on ℓ_p.

If instead of normalized, we assume a weaker condition that (f_n) is semi-normalized, that is there exists $C_1, C_2 > 0$ such that for all n, $C_1 \leq \|f_n\| \leq C_2$ then we can still do a similar computation to obtain that T is an isomorphism. $[f_n]$ is an isomorphic copy of ℓ_p in $L_p(\mu)$.

Lemma 37. *Let (x_n) be a basic sequence in X. The following are equivalent: (y_n) is a basic sequence equivalent to (x_n). There exists a surjective isomorphism $T : [x_n] \longrightarrow [y_n]$ such that $Tx_n = y_n$. (y_n) is basic and for every sequence (α_k) of scalars $\sum_{k=1}^{\infty} \alpha_k x_k$ converges iff $\sum_{k=1}^{\infty} \alpha_k y_k$ converges. There exists $C > 0$ such that for all $m, \alpha_1, \ldots, \alpha_m$ such that*

$$\frac{1}{C}\left\|\sum_{k=1}^{m} \alpha_k x_k\right\| \leq \left\|\sum_{k=1}^{m} \alpha_k y_k\right\| \leq C\left\|\sum_{k=1}^{m} \alpha_k x_k\right\|.$$

Proof. $i \implies iii$ is trivial. $iii \implies ii$, suppose we have iii, suppose (y_n) is basic. Suppose

$$\sum_{k=1}^{\infty} \alpha_k x_k$$

converges. This means

$$(\sum_{k=1}^{n} \alpha_k x_k)_n$$

converges, hence Cauchy, so

$$\left\|\sum_{k=n}^{m} \alpha_k x_k\right\| \longrightarrow 0$$

as $m, n \longrightarrow \infty$. By iii one has

$$\left\|\sum_{k=n}^{m} \alpha_k y_k\right\| \longrightarrow 0$$

as $m, n \longrightarrow \infty$. Two $ii \implies i$, we define $T : [x_n] \longrightarrow [y_n]$ via $T(\sum_{k=1}^{\infty} \alpha_k x_k) = \sum_{k=1}^{\infty} \alpha_k y_k$. By ii, T is well-defined, linear, and bounded by closed graph theorem. WLOG $[x_n] = X$ and $[y_n] = Y$, otherwise just replace $[x_n]$ with X and stuff. Suppose that $u_m \longrightarrow 0 \in X$ and $Tu_m \longrightarrow v \in Y$. We want to show that $v = 0$. Then $u_m = \sum_{k=1}^{\infty} \alpha_k^m x_k$. $v = \sum_{k=1}^{\infty} \beta_k y_k$. For each n we know that x_n^* is continuous, so $x_n^*(u_m) = \alpha_n^m \longrightarrow x_n^*(0) = 0$, hence, $\alpha_n^m \longrightarrow_m 0$ for each n. Then $Tu_m = T(\sum_{k=1}^{\infty} \alpha_k^n x_k) = \sum_{k=1}^{\infty} \alpha_k^m y_k$ but $\alpha_n^m = y_m^*(Tu_m) \longrightarrow y_n(v) = \beta_n$. This implies $\beta_m = 0$ for all m, so $v = 0$. By closed graph T is bounded, by definition T is onto, T is one-to-one. By Banach's theorem (open mapping), T is an isomorphism. □

Corollary 15. *Let (x_n) and (y_n) be two equivalent basic sequences (possibly in different spaces). If x_n is weakly null then y_n is weakly null.*

Proof. Let T be as in proposition. $Tx_n = y_n$. T is bounded, hence w-to-w continuous. $\square$

Recall, a seuqence (x_n) in a Banach space is absic if it is a basis of a closed usbspace. Equivalently it is a basis of $[x_n]$. Two basic sequences of (x_n) and (y_n) are equivalent $(x_n) \sim (y_n)$ if there is a surjective isomorphism $T : [x_n] \longrightarrow [y_n]$, $Tx_n = y_n$. Equivalently (x_n) and (y_n) have the same convergent series. Equivalently there is $C > 0$ and for all $\alpha_1, \ldots, \alpha_n$ such that

$$\frac{1}{C}\left\|\sum_{k=1}^{n} \alpha_k x_k\right\| \leq \left\|\sum_{k=1}^{n} \alpha_k x_k\right\| \leq C\left\|\sum_{k=1}^{n} \alpha_k x_k\right\|.$$

Every normalized or even seminormalized disjoint sequence in $L_p(\mu)$ is equivalent to the unit vector basis of ℓ_p for $1 \leq p < \infty$. We move onto an application now: Let $(r_n)_{n=1}^{\infty}$ be the Rademacher sequence without $r_0 = 1$. (r_n) is a block sequence of the Haar basis, so (r_n) is a basic sequence. For $t \in [0,1]$ consider its binary expansion $t = 0.b_1 b_2 b_3 \ldots$, $t = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$ for $b_k \in \{0,1\}$. This expansion is unique almost everywhere except a countable set. One sees that $r_k(t)$ is essentially the $k-$th binary digit at t. For each n, we can think of r_n as a random variable taken ± 1 with probability $1/2$. r_n's are independent. Fix n and consider $r_1, \ldots, r_n$. They generate a dyatdic partition of $[0,1]$ into 2^n equal subintervals. On these subintervals r_k's yield all possible choices of signs. Fix in addition scalars $\alpha_1, \ldots, \alpha_n$ and consider $f = \sum_{k=1}^{n} \alpha_k r_k$ on the first subinterval $f(t) = \alpha_1 + \ldots + \alpha_n$. On the second $f(t) = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} - \alpha_n$ and so on. We get all possible choices of $\sum_{\pm}(\pm\alpha_k)$ for all possible choices of sings with equal probabilities.

Theorem 38 (Khinchin's Inequality)**.** *For each $1 \leq p < \infty$ there exists $A_p, B_p > 0$ such that for all $\alpha_1, \ldots, \alpha_n$ then*

$$A_p \left(\sum_{k=1}^{\infty} |\alpha_k|^2\right)^{1/2} \leq \left\|\sum_{k=1}^{n} \alpha_k r_k\right\|_{L_p[0,1]} \leq B_p \left(\sum_{k=1}^{\infty} |\alpha_k|^2\right)^{1/2}$$

In particular, as a basic sequence (r_k) is equivalent to the unit vecotr basis of ℓ_2. In particular, there is an isomorphic embedding $T_p : \ell_2 \longrightarrow L_p[0,1]$ where $T_p e_n = r_n$ and $T_p\left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \sum_{k=1}^{\infty} \alpha_k r_k$. Let $\mathrm{Rad}_p = \mathrm{im} T_p = [r_n]$ in $L_p[0,1]$. Since T is an isomorphism $\mathrm{Rad}_p \cong \ell_2$ and hence $\mathrm{Rad}_p \cong \mathrm{Rad}_q$ for $p, q \in [1, \infty)$.

Lemma 39. Rad_p *does not depend on p as a set. On* Rad *all L_p norms are equivalent.*

Proof. See my pictures. $\square$

What happens when $p = \infty$? Then the rademacher sequence $r_n \in L_\infty[0,1]$ is equivalent to the unit vector basis of ℓ_1 and $\left\|\sum_{k=1}^{\infty} \alpha_k r_k\right\|_\infty = \sum_{k=1}^{\infty} |\alpha_k|$.

1.2 Complemented Subspaces

If X is a vector space and Y is a subspace then there exists a subspace $Z \subset X$ such that $X = Y \oplus Z$. That is, $X = Y + Z$ and $Y \cap Z = \{0\}$. Z is not unique. For each $x \in X$ there exists unique $y \in Y$ and $z \in Z$ such that $x = y + z$. Define $P : X \longrightarrow X$ by $P(x) = y$. Then $P : X \longrightarrow X$, $\text{im}P = Y$ and $\ker P = Z$. Then $P^2 = P$ and P is the projection onto Y along Z. $I - P$ is the projection onto Z along Y.

We want to consider closed subspaces now. Let Y be a closed subspace of X, it may be impossible to find a closed subspace Z of X such that $X = Y \oplus Z$. If such a closed subspace Z exists we say that Y is complemented in Z. If X is a Banach space then $X = Y \oplus Z$ for two linear subspacews. Let P be the corresponding projection. Then P is continuous iff both Y and Z are closed.

Proof. $Z = \ker P$ and $Y = \ker(I - P)$. We used the closed graph thoerem to $x_n \longrightarrow 0 \in X$. Then (Px_n) is in Y, Y is closed, so $y \in Y$. Then $x_n - Px_n = (I - P)x_n$ in Z. Then $x_n - Px_n \longrightarrow 0 - y = -y$ so $y \in Z$, so $y = 0$. □

Question, for every Banach space X are there two infinite dimensional spubspaces Y and Z such that $X = Y \oplus Z$ or in other words is X decomposable? Is every Banach space decomposable? No. GRowers and Mayrey found a banach space which is indecomposible, moreover every infinite dimensional dclosed subspace of it is also indecomposable. X is herediarily decomposible (that is what we call it).

Recall, if X is a Banach space and $X = Y \oplus Z$ the direct sum of subspaces, P is the projection onto Y along Z. Then P is continuous iff both Y and Z are closed. A closed subspaces Y of X is complemented if $X = Y \oplus Z$ for some closed subspace Z of X. Equivalently, Y is the range of a continuous projection. X/Y is again a Banach space but $X \cong Y \oplus (X/Y)$ but X/Y is not a subspace of X.

Example 31. In a Hilbert space, every closed subspace is the range of an orthogonal projection; hence complemented.

Example 32. Every finite dimensional subspace is complemented.

Example 33. If $X = L_p(\mu) = L_p(\Omega, F, \mu)$ let $\Omega \in F$. Then $Y = \{f \in X : \text{supp}f \subset \Omega\} \cong L_p(\Omega, \mu)$. Y is complemented, take $P : X \longrightarrow X$, $Pf = f\chi_\Omega$, a special case of (3) where we take $X = \ell_p$ or c_0 and $\Omega \subset \mathbb{N}$. For $x = (x_i)$ in X define $(Px)_i = \{x_i \text{ if } i \in \Omega\}$ and 0 otherwise. Then $Y = \text{im}(P) = \{x \in X : \text{supp}x \subset \Omega\}$.

Example 34. Let $X = L_p(\mu)$ for $1 \leq p < \infty$. (f_n) is a disjoint normalized sequence in X. We already know that (f_n) is a basic sequence, $(f_n) \sim$ uvb of ℓ_p. Then $[f_n] \cong \ell_p$. We claim that $[f_n]$ is complemented in X. The sketch is to find a dsjoint sequence of sets (A_n) such that $\text{supp}(f_n) \subset A_n$. $f_n \in L_p(A_n)$ and thus $L_p(A_n)^* = L_q(A_n)$ where $q = p^*$. Find $g_n \in L_q(A_n)$ where $\|g_n\|_q = 1 = \langle g_n, f_n \rangle$, we may view $g_n \in L_q(\Omega)$. Verify that for each

$f \in L_p(\mu)$ one has $\sum \langle f, g_n \rangle f_n$ converges in $L_p(\mu)$. Then denote the sum by Pf. $P : X \longrightarrow X$ is a bounded projection whose range is $[f_n]$.

What happens when $p = \infty$? then $[f_n] \cong c_0$. Is it complemented even in the special case of $X = \ell_\infty$?

Theorem 40. *c_0 is not complemented in ℓ_∞.*

Proof. Skip, but interesting to see. $\square$

What is the space ℓ_∞ / c_{00}, it is precisely those seuqences $x_n \sim y_n$ such that $x_n - y_n \longrightarrow 0$

If $Y \subset Z \subset X$ is complemented in X then it is complemented in Z.

Theorem 41. *X^* is complemented in X^{***}.*

Proof. $\iota : X \longrightarrow X^{**}$ and $k : X^* \longrightarrow X^{***}$ the canonical inclusions. Then $X^{***} \longrightarrow X^* \longrightarrow X^{***}$ is a projection map with the range being X^*. $\square$

Theorem 42. *For $1 < p < \infty$. Rad is complemented in $L_p[0,1]$. For $p = 2$. Trivial, $L_2[0,1]$ is a Hilbert space. Every closed subspace is complemented. (r_n) is anorhtonormal basis for Rad_2 and we can explicitly write the orthogonal projection from $L_2[0,1]$ onto Rad.*

$$Pf = \sum_{n=1}^{\infty} \langle f, r_n \rangle r_n.$$

For $2 \le p < \infty$ $L_p[0,1]$ is a subset of $L_2[0,1]$ and $\|\cdot\|_2 \le \|\cdot\|_p$. One has the maps

$$L_p \longrightarrow L_2 \longrightarrow \mathrm{Rad}_2 \cong \mathrm{Rad}_p.$$

This gives us a continuous projection from L_p onto Rad_p. When $1 < p < 2$ let $f, g \in L_2$ then we can write $\langle Pf, g \rangle = \sum_{m=1}^{\infty} \langle f, r_n \rangle \langle g, r_n \rangle = \langle f, Pg \rangle = \int fPg\, dx$. Let $q = p^*$. Then $2 < q < \infty$. $L_q \subset L_2 \subset L_p$. Let $f \in L_2$ and $g \in L_q$. Then $g \in L_2$ so $\langle Pf, g \rangle \le \|f\|_p \|g\|_q$. Then we know that $P : L_q \longrightarrow L_q$ is bounded and say $C = \|P\|$ so $\langle Pf, g \rangle \le \|f\|_p C \|g\|$ and thus $\|Pf\|_p = \sup \langle Pf, g \rangle \le C \|f\|_p$. For all $f \in L_2$ $\|Pf\|_p \le C \|f\|_p$. L_2 is dense in L_p so P extends to a continuous operator $\tilde{P} : L_p \longrightarrow L_p$. $\tilde{p}$ is still a projection and $\mathrm{im}P = \mathrm{Rad}_2 = \mathrm{Rad}_p$.

Theorem 43 (Mazur)**.** *Every Banach space has a basic sequence. That is, it has a closed subspace with a basis.*

The idea is in a Hilbert space we can pick $e_1 \neq 0$ and find e_2 orthogonal to e_1, find e_3 orthogonal to $\mathrm{span}(e_1, e_2)$. This yields an orthogonal sequence, hence a basis.

Lemma 44. *Let Y be a finite dimensional subspace of X. Let $\varepsilon > 0$. Then there exists non-zero $x \in X$ such that for all $y \in S_Y$ the straight line $\{y + \lambda x : \lambda \in F\}$ does not meer $(1-\varepsilon)B_X$.*

Proof. Since Y is finite dimensional, S_Y is compact so we can find an epsilon net $\{y_1, \ldots, y_m\}$ for S_Y. For each $i = 1, \ldots, m$ by Hahn-Banach we can find a functional $f_i \in S_{X^*}$ such that $f_i(y_i)$. Then $\bigcap \ker f_i$ is of finite codimension, so non-trivial. Take any non-zero $x \in \bigcap \ker f_i$. Then $f_i(X) = 0$ for all i. Let yS_Y. Find i such that $\|y - y_i\| < \varepsilon$. So for every $\lambda \in F$ one then has $1 = f_i(y_i) = f_i(y_i + \lambda x) \leq \|f_i\| \|y_i + \lambda x\| \leq \|y_i - y\| + \|y + \lambda x\| \leq \varepsilon + \|y + \lambda x\|$. This means $\|y + \lambda x\| \geq 1 - \varepsilon$. □

This means for all $y \in Y$ $\|y + \lambda x\| \geq (1 - \varepsilon) \|y\|$.

Theorem 45 (Mazur)**.** *For every $K > 1$ every Banach space X contains a basic sequence with basis constant $\leq K$.*

Proof. Fix $K > 1$. Let $\varepsilon_n \longrightarrow 0$ so fast that $\prod_{n=1}^{\infty}(1 - \varepsilon_n) > 1/k$. Take any non-zero x_1. Put $Y = \operatorname{span}(x_1)$ and apply Lemma to Y and ε_1 to get x_2. Put $Y_2 = \operatorname{span}(x_1, x_2)$. Iterate. Claim is that $\{x_n\}$ is basic with basis constant $\leq K$. It suffices to show the basis inequality. Let $n < m$ and $\alpha_1, \ldots, \alpha_m \in F$. Then $\|\alpha_1 x_1 + \ldots + \alpha_m x_m\| \geq (1 - \varepsilon_m) \|\alpha_1 x_1 + \ldots + \alpha_{m-1} x_{m-1}\| \geq (1 - \varepsilon_m)(1 - \varepsilon_{m-1}) \cdot \ldots \cdot (1 - \varepsilon_{n+1}) \|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq 1/K \|\alpha_1 x_1 + \ldots + \alpha_n x_n\|$. □

Theorem 46. *Let $S \subset X$ such that $0 \in \overline{S}^w \setminus \overline{S}$. Then for every $K > 1$, S contains a basic sequence iwth constant $\leq K$.*

We skip proof.

Corollary 16. *If (x_n) converges to zero weakly but not in norm then it has a basic subsequence.*

Proof. Passing to a subsequence, x_n is bounded below. Take $S = \{x_n\}$ and apply the above. □

1.3 Unconditionally Convergent Series

Theorem 47. *Given a sequence $(x_n) \subset X$. The following are equivalent: The sum*

$$\sum_{n=1}^{\infty} \alpha_n x_n$$

converges for any bounded (α_n).

$$\sum_{n=1}^{\infty} \alpha_n x_n$$

converges for α_n such that $|\alpha_n| = 1$.

$$\sum_{n=1}^{\infty} \pm x_n$$

converges.

$$\sum_{k=1}^{\infty} x_{nk}$$

converges.

$$\sum_{n=1}^{\infty} x_{\sigma(n)}$$

converges for every bijection σ.
For all $\varepsilon > 0$ *there is* N *such that for all finite* A *with* $\min A \geq A$ *one has*

$$\left\| \sum_{n \in A} x_n \right\| < \varepsilon.$$

$$\sum_{n=1}^{\infty} |f(x_n)|$$

converges for f *in* S_{X^*}.

Again given this we say that x_n converges unconditionally.

Remark 1. *Suppose* $\sum_{k=1}^{\infty} x_k$ *converges unconditionally. Then the set*

$$\{sum_{k=1}^{\infty} \alpha_k x_k : |\alpha_k| \leq 1\}$$

is norm bounded.

Indeed, take $\varepsilon = 1$. We can find n such that for all $m > n$ and all $f \in B_{X^*}$ one has

$$\sum_{k=n}^{m} |f(x_k)| < 1.$$

Let (α_k) be such that $|\alpha_k| \leq 1$ for all k. Then by Hahn-Banach there exists $g \in S_{X^*}$ such that

$$\left\| \sum_{k=n}^{m} \alpha_k x_k \right\| = \left| f\left(\sum_{k=n}^{m} \alpha_k x_k \right) \right| \leq \sum_{k=n}^{m} |f(x_k)| \leq 1$$

so letting $m \longrightarrow \infty$ we obtain

$$\left\| \sum_{k=n}^{\infty} \alpha_k x_k \right\| \leq 1$$

. Also,

$$\left\| \sum_{k=1}^{n-1} \alpha_k x_k \right\| \leq \sum_{k=1}^{n-1} \|x_k\|.$$

In particular this means that

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| \leq \sum_{k=1}^{n} \|x_k\| + 1.$$

Remark 2. *In (v) we say that every permutation converges. Further, the sums will be all the same (hint, use 6).*

Definition 35. A basis (e_k) is said to be unconditional if the basis expansions converge unconditionally for every $x \in X$, we know that

$$x = \sum_{k=}^{\infty} \alpha_k e_k$$

is unique. But further we require in addition that this series converges unconditionally.

Example 35. (e_k) =uvb of ℓ_p or c_0 $(1 \leq p < \infty)$ then $\sum |\alpha_k|^p$ converges iff $\sum |\pm\alpha_k|^p$ converges.

Let (e_k) be an unconditional basis. Fix $\lambda = (\lambda_k) \in \ell_\infty$ and suppose $\|\lambda\|_\infty = 1$. Let $x \in X$. Expand $x = \sum_{k=1}^{\infty} \alpha_k e_k$. This series converges unconditionally. Then

$$\left\| \sum_{k=1}^{\infty} \lambda_k \alpha_k e_k \right\|$$

also converges. Denote this sum by $T_\lambda x$. Clearly T is linear. Thus, we can view X as a sequence space and T_λ is a diagonal operator. We know that T is well defined by what we had done before. Not only is it well defined, but we claim also that it is bounded. Here we use closed graph theorem. If $x^{(m)} \longrightarrow 0$ and $Tx^{(m)} \longrightarrow u$ then $x^{(m)} = \sum_{k=1}^{\infty} \alpha_k^{(m)} e_k$ and so $Tx^{(m)} = \sum_{k=1}^{\infty} \lambda_k \alpha_k^{(m)} e_k$ and $u = \sum_{k=1}^{\infty} \beta_k e_k$. Then e_k^* is continuous for each k so $x^{(m)} \longrightarrow 0$ implies $\alpha_k^{(m)} = e_k^*(x^{(m)}) \longrightarrow 0$ so $\alpha_k^{(m)} \longrightarrow 0$. Then $Tx^{(m)} \longrightarrow u$ implies $e_k^*(Tx^{(m)}) \longrightarrow e_k^*(u)$ which implies $\lambda_k \alpha_k^{(m)} \longrightarrow \beta_k$ and so $\beta_k = 0$ and so $u = 0$.

Further, we claim that $\{T_\lambda : \lambda \in B_{\ell_\infty}\}$ is uniformly bounded. This is just use uniform boundedness-principle. If we denote $M = \sup_\lambda \|T_\lambda\|$ we say that M is the unconditional basis constant of (e_k).

If (e_k) is an unconditional basis with basis constant M and then every bounded seqeuence yields (λ_k) yields a bounded operator $\|T_\lambda\| \leq M \|\lambda\|_\infty$. For a special case, for $\lambda \in \ell_\infty$ where λ_k's are zeros and ones then we define $\lambda_i = 1$ if $i \in A$ and 0 o/w. Then $T_\lambda = P_A$ where $P_A e_k = e_k$ if $k \in A$ and 0 otherwise. One has

$$P_A\left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \sum_{k \in A} \alpha_k e_k.$$

Then P_A is a projection where $\|P_A\| \leq M$. In particular X has many projections and any complemented subspaces. For example if $[e_{2k}] = \text{Range} P_A$ where $A = 2\mathbb{N}$ is a complemented subspace.

Let X be a Banach space with a Schauder basis. We view X as a sequence space and can order X coordinate wise saying $\sum \alpha_k e_k \leq \sum \beta_k e_k$ if $\alpha_k \leq \beta_k$

for all k. This is a partial order and linear in the sense that if $x \leq y$ then $\lambda x \leq \lambda y$for all $\lambda \geq 0$ and $x+z \leq y+z$ for all z. In general, this is not a lattice order and it does not respect the norm structure. Suppose in addition that we have a 1-unconditional basis (e_k). Then $x \vee y = \sum \max\{\alpha_k, \beta_k\} e_k$ and $x \wedge y$ is similar. X becomes a vector lattice under $|x| = \sum_{k=1}^{\infty} |\alpha_k| e_k = \sum_{k=1}^{\infty} \lambda_k \alpha_k e_k$ where $\lambda_k = |\alpha_k| / \alpha_k$ and moreover is a Banach lattice. Moreover, $\||x|\| = \|x\|$ and the norm is monotone. X then becomes a Banach lattice.

1.4 Bits and Pieces on Classical Banach Spaces

Recall from Hahn-Banach theorem that if X is a normed space and Y is a subsapce where $f : Y \longrightarrow k$ is linear and bounded. We can extend it to $\hat{f} : X \longrightarrow k$ where $\left\|\hat{f}\right\| = \|f\|$. From Math 418 we can replace k with $\ell_\infty = \ell_\infty(\Gamma)$ where Γ is any set.

Now we ask questions about universal spaces. Let X be a Banach space. We know that X embeds into X^{**} and thus $K = B_{X^*}$ is weak* compact. Consider $\hat{x} \in X^{**}$ restricted to B_{X^*}. The map $S : X \longrightarrow C(K)$ given by $x \longrightarrow \hat{x}_{|B_{X^*}}$ is an isometric embedding. This means that every Banach space is a closed subspace of some $C(K)$ space, where K depends on X. If we assume in addition that X is seperable then we proved at some point that (B_{X^*}, w^*) is metrizable. A result from topology states that every compact metric space is a continuous image of the Cantor set Δ. There exists a continuous map $F : \Delta \longrightarrow K$ which is onto. This induces an isometry $C(K) \longrightarrow C(\Delta)$ given by $f \longrightarrow f \circ F$. This means that the new map $X \longrightarrow C(\Delta)$ is an isometric embedding into $C(\Delta)$, so now the dependence on K is not needed. Using the fact that $\Delta \subset [0,1]$ is dense in $[0,1]$ we get that $X \longrightarrow C[0,1]$ by continuous extension. Thus every seperable Banach space is a closed subspace of $C[0,1]$. In some sense $C[0,1]$ is a "universal" seperable Banach space.

A related fact is that ℓ_∞ contains every seperable Banach space (but is itself not seperable).

Proof. Let X be a seperable Banach space. Then at some point we showed that B_{X^*} is weak* seperable. Let (f_n) be a dense sequence in B_{X^*} define $T : X \longrightarrow \ell_\infty$ by $Tx = (f_n(x))$. This map is linear.

$$\|Tx\| = \sup_n |f_n(x)| = \sup_{f \in B_{X^*}} |f(x)| = \|x\|.$$

Thus, $Tx \in \ell_\infty$ and T is an isometric embedding. $\square$

Quotients of ℓ_1: We start with some preliminaries

Definition 36. Open Mapping Lemma from Math 418. Let $T : X \longrightarrow Y$. If $B_Y^\circ \subset \overline{TB_X^\circ}$ then $B_Y^\circ \subset TB_X^\circ$.

Suppose that Y is a quotient space of X, that is $Y = X/Z$ for some closed subspace Z of X. Definte $Q : X \longrightarrow Y$ be the quotient map. Then $QB_X^\circ = B_Y^\circ$, an easy exercise. Conversely suppose that X and Y are Banach spaces and suppos that there exists a $T \in L(X,Y)$ such that $TB_X^\circ = B_Y^\circ$. Then Y is a quotient space of X. Another exercise (hint take $Z = \ker T$). We need one more fact, that if X is a Banach space and $(x_n) \subset B_X$ is a sequence. Then we can define a bounded linear operator $T : \ell_1 \longrightarrow X$ by $Te_n = x_n$, by linearity extend to the span, and then $\|T\sum \alpha_i e_i\| = \|\sum \alpha_i x_i\| \leq \sum |\alpha_i| = \|\sum \alpha_i e_i\|_{\ell_1}$ so T is bounded on c_{00} so extends to a bounded operator $T : \ell_1 \longrightarrow X$.

Theorem 48. *Every seperable Banach space is isometrically isomorphic to a quotient of ℓ_1.*

Proof. Let X be a seperable Banach space and (x_n) be a dense sequence in B_X. As before we find an operator $T : \ell_1 \longrightarrow X$ such that $Te_n = x_n$. This operator has norm at most 1. This means that $TB_{\ell_1}^\circ \subset B_X^\circ$. On the other hand $B_X^\circ \subset B_X = [x_n] \subset \overline{TB_{\ell_1}} = \overline{TB_{\ell_1}^\circ}$. This means that $B_X^\circ = TB_{\ell_1}^\circ$ and so by X is a quotient of ℓ_1. $\square$

1.5 Uniform convexity and smoothness

Let X be a Banach space. We say that X is strictly convex if $\|x+y\| < \|x\| + \|y\|$ as long as x is not in the span of y. X is strictly convex iff every two dimensional subspace of X is strictly convex. We can reduce the definition to $x, y \in S_X$ and therefore all two dimensional spheres.

Lemma 49. *X is strictly convex iff for all $x, y \in S_X$ $x \neq y \implies \left\|\frac{x+y}{2}\right\| < 1$.*

Proof. for $x, y \in S_X$ $\|x+y\| < \|x\| + \|y\| = 2$ so $\left\|\frac{x+y}{2}\right\| < 1$. Now for the converse if we assume by contradiction that there exists x, y which are not in each other span but $\|x+y\| = \|x\| + \|y\|$. Then WLOG $0 < \|x\| \leq \|y\|$. Then

$$\|x/\|x\| + y/\|x\|\| \leq \|x/\|x\| + y/\|y\|\| + \|y/\|x\| - y/\|y\|\|,$$

where the first factor on the right is $\geq \|x/\|x\| + y/\|y\|\| - \|y/\|x\| - y/\|y\|\| = \|x+y\|/\|x\| - \|y\|(1/\|x\| - 1/\|y\|)$. This is precisely

$$\frac{\|x\| + \|y\|}{\|x\|} - \frac{\|y\|}{\|x\|} + \frac{\|y\|}{\|y\|} = 2,$$

which is a contradiction. $\square$

X is strictly convex iff S_X contains no straight line segments iff $\mathrm{ext} B_X = S_X$. ℓ_1^2 and ℓ_∞^2 are not strictly convex and thus since failure goes up this means that neither can $L_1(\mu)$ or $L_\infty(\mu)$ or c_0 or ℓ_∞.

If $x \in S_X$ then by Hahn-Banach Theorem there is $f \in S_{X^*}$ such that $f(x) = 1$. If $x \in S_X$ then we say that x is exposed if there exists $f \in S_{X^*}$ such that x is the only point of S_X such that $f(x) = 1$.

Lemma 50. *Every exposed point is extreme.*

Proof. Let $x \in S_X$ suppose that x is exposed but not extreme. Then $x = \alpha y + \beta z$ where $x \neq y \neq z \in S_X$ and $\alpha + \beta = 1$, $\alpha, \beta \geq 0$. Since x is exposed let f be as above. Then $1 = f(\alpha y + \beta z) = \alpha f(y) + \beta f(z)$, where $f(y) < 1$ and $f(z) < 1$ so $\alpha f(y) + \beta f(z) < \alpha + \beta = 1$, so $1 < 1$. □

Corollary 17. *X is strictly convex iff every point of S_X is exposed.*

Proof. Exposed implies extreme implies $S_X = \text{ext} B_X$. On the other hand if X is strictly convex but not exposed then there exists $x \in S_X$ which is not exposed. Take any $f \in S_{X^*}$ such that $f(x) = 1$ by Hahn-Banach. Since x is not exposed there exists $y \in S_X$ such that $f(y) = 1$. Then $f((x+y)/2) = \frac{f(x)}{2} + \frac{f(y)}{2} = 1$ so $\|(x+y)/2\| \geq 1$ so $\|(x+y)/2\| = 1$, a contradiction to strict convexity. □

Recall, we say that X is uniformly convex if for all $\varepsilon \in (0,2)$ and there is $\delta > 0$ and $x, y \in S_X$ if $\|x - y\| \geq \varepsilon$ then

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

It is sufficient to consider $x, y \in B_X$ and replace $\geq \varepsilon$ with $= \varepsilon$. Uniformly convex implies strictly convex, while in the finite dimensional case the two concepts agree. Often we take $\delta = \inf\{1 - \|x + y\| \, 2 : x, y \in S_X : \|x - y\| = \varepsilon\}$. This is continuous on $S_X \times S_X$ so if X is finite dimensional then the infimum is actually attained. X is uniformly convex iff for all sequences (x_n) and (y_n) in B_X if

$$\left\| \frac{x_n + y_n}{2} \right\| \longrightarrow 1$$

then $\|x_n - y_n\| \longrightarrow 0$.

Theorem 51 (Radon Riesz)**.** *If X is uniformly convex and $x_n \longrightarrow_w x$ and $\|x_n\| \longrightarrow \|x\|$ then $x_n \longrightarrow_{\|\cdot\|} x$*

Proof. If $x = 0$, done. So wlog $x \neq 0$ so $\|x\| = 1$. Then $x_n / \|x_n\| - x_n \longrightarrow_{\|\cdot\|} 0$. Replacing x_n with $x_n / \|x_n\|$ we may assume $\|x_n\| = 1$ for all n. Find $f \in S_X$ such that $f(x) = 1$. Then

$$f\left(\frac{x_n + x_m}{2}\right) \leq \|f\| \left\| \frac{x_n + x_m}{2} \right\| \leq 1.$$

On the other hand,

$$\frac{f(x_n) + f(x_m)}{2} \longrightarrow \frac{f(x) + f(x)}{2} = 1.$$

This means that

$$\left\| \frac{x_n + x_m}{2} \right\| \longrightarrow 1,$$

so $\|x_n + x_m\| \longrightarrow 0$ by uniform convexity. This means x_n is Cauchy, so converges, but then by uniqueness of limits we get norm convergence. □

Corollary 18. *If X is uniformly convex then X has the Kadec Klee property, i.e., the norm and weak topology agree on S_X.*

Proof. Every uniformly convex Banach space is reflexive $\square$

Proof. If $\xi \in X^{**}$ then we want to show that $\xi = \hat{x}$ for some $x \in X$. WLOG $\|\xi\| = 1$. Then Goldstein's theorem says that B_X is $w-^*$ dense in $B_{X^{**}}$ so we find (x_α) in B_X such that $\hat{x}_\alpha \longrightarrow_{w^*} \xi \in X^{**}$. Then

$$1 = \|\xi\| \leq \liminf \|\hat{x}_\alpha\| = \liminf \|x_\alpha\|$$

so $\|x_\alpha\| \longrightarrow 1$. Fix $\varepsilon > 0$ and find δ for uniform convexity. Then find $f \in S_{X^*}$ such that $\xi(f) > 1 - \delta$. Then for every α, β we have

$$f\left(\frac{x_\alpha + x_\beta}{2}\right) = \frac{f(x_\alpha) + f(x_\beta)}{2} = \frac{\hat{x}_\alpha(f) + \hat{x}_\beta(f)}{2} \longrightarrow \xi(f) > 1 - \delta.$$

Then there exists α_0 such that for all $\alpha, \beta \geq \alpha_0$ one has

$$1 - \delta < f((x_\alpha + x_\beta)/2) \leq \|(x_\alpha + x_\beta)/2\|$$

and thus by the definition of δ this means $\|x_\alpha - x_\beta\| < \varepsilon$. Thus x_α is Cauchy, so convergent to some $x_\alpha \longrightarrow_{\|\cdot\|} x$ and thus $\hat{x}_\alpha \longrightarrow_{w^*} \hat{x}$ so $\xi = \hat{x}$. $\square$

Here is an example of a Banach space which is strictly convex but not uniformly convex. Take $X = \ell_1$ and equip X with the norm $\|\cdot\| = \|\cdot\|_1 + \|\cdot\|_2$ which is defined since $\ell_1 \subset \ell_2$. Then $\|\cdot\|_1 \leq \|\cdot\| \leq 2\|\cdot\|$. Thus $X \cong \ell_1$ and since ℓ_1 is not reflexive neither can X be. X is strictly convex since

$$\|x + y\| = \|x + y\|_1 + \|x + y\|_2 \leq \|x_1\| + \|y\|_1 + \|x + y\|_2 < \|x\| + \|y\|,$$

by the strict convexity of ℓ_2.

Our goal is to show that if $1 < p < \infty$ then $L_p(\mu)$ is uniformly convex. The case $p = 2$ is trivial since L_2 is a Hilbert space and in a Hilbert space we have the parallelogram law. For $p > 2$.

Lemma 52. *For all real $a, b \in \mathbb{R}$, $|a + b|^p + |a - b|^p \leq 2^{p-1}(|a|^p + |b|^p)$.*

Proof. One has

$$(|a + b|^p + |a - b|^p)^{1/p} \leq (|a + b|^2 + |a - b|^2)^{1/2} = (2(|a|^2 + |b|^2))^{1/2} = 2^{1/2}\left(|a|^2 + |b|^2\right)^{1/2},$$

Then by generalized Holders inequality

$$\leq 2^{1/2}\left|1^r + 1^r\right|^{1/r}(|a|^p + |b|^p)^{1/p} = 2^{1/2}2^{1/2-1/p}(|a|^p + |b|^p)^{1/p}.$$

$\square$

Corollary 19. *If $2 < p < \infty$ and $f, g \in L_p(\mu)$ then*

$$(\|f+g\|^p + \|f-g\|^p)^{1/p} \leq 2^{1-1/p}(\|f\|^p + \|g\|^p)^{1/p}.$$

Corollary 20. *If $2 < p < \infty$ then $L_p(\mu)$ is uniformly convex.*

Proof. If $f, g \in S_{L_p(\mu)}$ with $\|f-g\| = \varepsilon$ then $\|f+g\|^p + \varepsilon^p \leq 2^{p-1}(1+1) = 2^p$. Then $\|f+g\|^p \leq 2^p - \varepsilon^p$ and so $\|(f+g)/2\|^p \leq 1 - (\varepsilon/2)^p := 1 - \delta$. □

Now we visit the case $1 < p < 2$. Take $q = \frac{p}{p-1} = \frac{1}{1-1/p}$. Then $q > 2$ and $q - 1 = \frac{1}{p-1}$ so $(p-1)(q-1) = 1$.

Lemma 53. *For $x \in [0,1]$, $(1+x)^q + (1-x)^q \leq 2(1+x^p)^{q-1}$.*

Proof. For $x, \alpha \in [0,1]$ put $f(\alpha, x) = (1+\alpha^{1-q}x)(1+\alpha x)^{q-1} + (1-\alpha^{1-q}x)(1-\alpha x)^{q-1}$. Then $f(1,x) = LHS$ and $f(x^{p-1}, x) = RHS$. It suffices to prove that $f(1,x) \geq f(x^{p-1}, x)$. □